2/8/2002, On the solutions of linear systems — Math 21b

HOMEWORK: 1.3:4,14,26*,34,48,50, **Due:** Tue 2/12/2002

COEFFICIENT MATRIX AND AUGMENTED MATRIX. Linear system

$$\left| \begin{array}{ll} 3x - y - z & = 0 \\ -x + 2y - z & = 0 \\ -x - y + 3z & = 9 \end{array} \right|$$

Rewrite it as $A\vec{x} = \vec{b}$, with the **coefficient matrix** A and vectors $\mathbf{b}, \mathbf{x}$.

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}$$

The **augmented matrix** is

$$B = \left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & 9 \end{array} \right].$$

GAUSS-JORDAN ELIMINATION. Gauss-Jordan elimination is an algorithm where successive subtraction of multiples of other rows or scaling brings the matrix into **reduced row echelon form**.
The elimination process uses three possible steps, the **elementary row operations**:
- Swap two rows.
- Divide a row by a scalar
- Subtract a multiple of a row from an other row.

The process transfers a given matrix A into a new matrix $\text{rref}(A)$ in reduced row echelon form.

REDUCED ECHELON FORM. A matrix is called in **reduced row echelon form**

1) If a row has nonzero entries, then the first nonzero entry is 1. (**leading one**)
2) If a column contains a leading 1, then the other column entries are 0.
3) If a row has a leading 1, then every row above has leading 1's to the left.

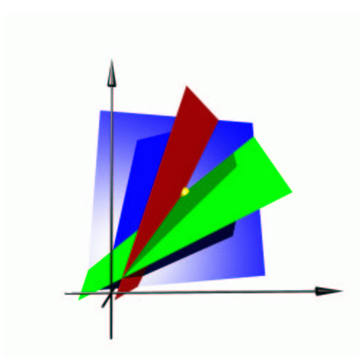
RANK. The number of leading 1 in $\text{rref}(A)$ is called the rank of A.

SOLUTIONS OF LINEAR EQUATIONS. If $A\vec{x} = \vec{b}$ is a linear system of equations with m equations and n unknowns, then A is a $m \times n$ matrix.
The reduced echelon form of the augmented matrix B determines on how many solutions the linear system $Ax = b$ has. We have the following three possibilities:

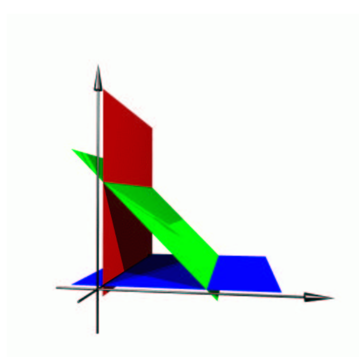
- **Exactly one solution.** There is a leading 1 in each row but not in the last row.
- **No solutions.** There is a leading 1 in the last row.
- **Infinitely many solutions.** There are rows without leading 1 and no leading 1 is in the last row.

If $m < n$ (there are less equations then unknows), then we have either zero or infinitely many solutions.

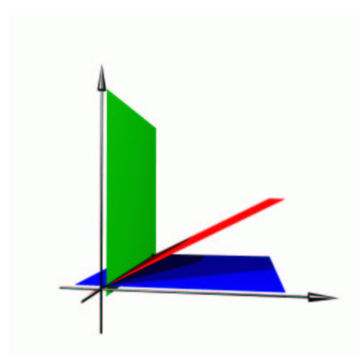
(**exactly one solution**)



(**no solution**)



(**infinitely many solutions**)



EXAMPLES.

$$\begin{bmatrix} 0 & 1 & 2 & | & 2 \\ 1 & -1 & 1 & | & 5 \\ 2 & 1 & -1 & | & -2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 1 & | & 5 \\ 0 & 1 & 2 & | & 2 \\ 2 & 1 & -1 & | & -2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 1 & | & 5 \\ 0 & 1 & 2 & | & 2 \\ 0 & 3 & -3 & | & -12 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 1 & | & 5 \\ 0 & 1 & 2 & | & 2 \\ 0 & 1 & -1 & | & -4 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 3 & | & 7 \\ 0 & 1 & 2 & | & 2 \\ 0 & 0 & -3 & | & -6 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 3 & | & 7 \\ 0 & 1 & 2 & | & 2 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$$

Rank$(A) = 3$, Rank$(B) = 3$.

$$\begin{bmatrix} 0 & 1 & 2 & | & 2 \\ 1 & -1 & 1 & | & 5 \\ 1 & 0 & 3 & | & -2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 1 & | & 5 \\ 0 & 1 & 2 & | & 2 \\ 1 & 0 & 3 & | & -2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 1 & | & 5 \\ 0 & 1 & 2 & | & 2 \\ 0 & 1 & 2 & | & -7 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 1 & | & 5 \\ 0 & 1 & 2 & | & 2 \\ 0 & 0 & 0 & | & -9 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 3 & | & 7 \\ 0 & 1 & 2 & | & 2 \\ 0 & 0 & 0 & | & -9 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 3 & | & 7 \\ 0 & 1 & 2 & | & 2 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$$

Rank$(A) = 2$, Rank$(B) = 3$.

$$\begin{bmatrix} 0 & 1 & 2 & | & 2 \\ 1 & -1 & 1 & | & 5 \\ 1 & 0 & 3 & | & 7 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 1 & | & 5 \\ 0 & 1 & 2 & | & 2 \\ 1 & 0 & 3 & | & 7 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 1 & | & 5 \\ 0 & 1 & 2 & | & 2 \\ 0 & 1 & 2 & | & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 1 & | & 5 \\ 0 & 1 & 2 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 3 & | & 7 \\ 0 & 1 & 2 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Rank$(A) = 2$, Rank$(B) = 2$.

VECTOR AND MATRIX FORM OF A LINEAR EQUATION.

$$A\vec{x} = \begin{bmatrix} -\vec{w}_1- \\ -\vec{w}_2- \\ \dots \\ -\vec{w}_m- \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \dots \\ \vec{w}_m \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{b}\,.$$

IN WORDS.
1) The entries b_i are the dot product of row vectors $\vec{w}_i$ with $\vec{x}$.
2) The vector $\vec{b}$ is a sum of scaled column vectors $\vec{v}_j$ of A. (Linear combination).

MATRIX "JARGON". A rectangular array of numbers is called a **matrix**.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

If the matrix has m **rows** and n **columns**, it is called a $m \times n$ matrix. A matrix with one column only is called a **column vector**. The entries of a matrix are denoted by a_{ij}, where i is the row number and j is the column number.

In the case of the linear equation above, the matrix A is a square matrix and the augmented matrix B above is a 3×4 matrix.

Matrices can be added, subtracted if they have the same size:

$$A+B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}+b_{m2} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}.$$

They can also be scaled by a scalar λ:

$$\lambda A = \lambda \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix}$$