The arc length is independent of the parameterization of the curve.

If \( t \in [a, b] \mapsto \mathbf{r}(t) \) with velocity \( \mathbf{v}(t) = \mathbf{r}'(t) \) and speed \( |\mathbf{v}(t)| \), then \( \int_a^b |\mathbf{v}(t)| \, dt \) is called the arc length of the curve. For space curves this is

\[
L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt
\]

**Parameter Independence.** The arc length is independent of the parameterization of the curve.

**Reason.** Changing the parameter is a change of variables (substitution) in the integration.

**Example.** The circle parameterized by \( \mathbf{r}(t) = (\cos(t^2), \sin(t^2)) \) on \( t = [0, \sqrt{2\pi}] \) has the velocity \( \mathbf{r}'(t) = 2t(-\sin(t), \cos(t)) \) and speed \( 2t \). The arc length is \( \int_0^{\sqrt{2\pi}} 2t \, dt = t^2|_0^{\sqrt{2\pi}} = 2\pi. \)

**Remark.** Often, there is no closed formula for the arc length of a curve. For example, the Lissajous figure \( \mathbf{r}(t) = (\cos(3t), \sin(5t)) \) has the length \( \int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)} \, dt \). This integral must be evaluated numerically. If you do the Mathematica Lab, you will see how to do that with the computer.

**The Material Below Is Not Part of This Course.**

**Curvature.**

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \text{ unit tangent vector}
\]

\[
\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)| \text{ curvature}
\]

**Example. Circle**

\[
\mathbf{r}(t) = (r \cos(t), r \sin(t)).
\]

\[
\mathbf{T}(t) = (-\sin(t), \cos(t)),
\]

\[
|\mathbf{T}'(t)| = r.
\]

**Example. Helix**

\[
\mathbf{r}(t) = (\cos(t), \sin(t), t).
\]

\[
\mathbf{T}(t) = (-\sin(t), \cos(t), 1),
\]

\[
|\mathbf{T}'(t)| = \sqrt{2}.
\]

**Interpretation.**

If \( s(t) = \int_0^t |\mathbf{r}'(t)| \, dt \), then \( s'(t) = ds/dt = |\mathbf{r}'(t)|. \) Because we have

\[
|d\mathbf{T}|/ds = |\mathbf{T}'(t)|/|\mathbf{r}'(t)| = \kappa(t).
\]

"The curvature is the length of the acceleration vector if \( \mathbf{r}(t) \) traces the curve with constant speed 1."

A large curvature at a point means that the curve is strongly bent. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration.

<table>
<thead>
<tr>
<th>Small curvature</th>
<th>Large curvature</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa = 1/r )</td>
<td>( \kappa = 1/r = 2 )</td>
</tr>
</tbody>
</table>
The curve $\mathbf{r}(t) = (t, f(t))$, which is the graph of a function $f$ has the velocity $\mathbf{v}(t) = (1, f'(t))$ and the unit tangent vector $\mathbf{T}(t) = \frac{1}{\sqrt{1 + f'(t)^2}}$ and after some simplification
\[
\kappa(t) = \frac{1}{\sqrt{1 + f'(t)^2}}
\]

EXTRACTION.
\[
\begin{align*}
\mathbf{r}(t) &= (t, f(t)), \\
\mathbf{v}(t) &= (1, f'(t)), \\
\mathbf{T}(t) &= \frac{1}{\sqrt{1 + f'(t)^2}}, \\
\kappa(t) &= \frac{1}{\sqrt{1 + f'(t)^2}}.
\end{align*}
\]

TANGENT/NORMAL/BINORMAL.
\[
\begin{align*}
\mathbf{T}(t) &= \mathbf{r}'(t) \\
\mathbf{N}(t) &= \mathbf{T}'(t) \\
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t)
\end{align*}
\]

Because $\mathbf{T}(t) \cdot \mathbf{T}(t) = 1$, we get after differentiation $\mathbf{T}'(t) \cdot \mathbf{T}(t) = 0$ and $\mathbf{N}(t)$ is perpendicular to $\mathbf{T}(t)$.

WHERE IS CURVATURE NEEDED?

OPTICS. If a curve $\mathbf{r}(t)$ represents a wavefront and $\mathbf{n}(t)$ is a unit vector normal to the curve at $\mathbf{r}(t)$, then $\mathbf{s}(t) = \mathbf{r}(t) + \mathbf{n}(t)/\kappa(t)$ defines a new curve called the caustic of the curve. Geometers call that curve the evolute of the original curve.

HISTORY.

Aristotle: (350 BC) distinguishes between straight lines, circles and "mixed behavior"
Oresme: (14'th century): measure of twist called "curvitas"
Kepler: (15'th century): circle of curvature.
Huygens: (16'th century): evolutes and involutes in connection with optics.
Newton: (17'th century): circle has constant curvature inversely proportional to radius. (using infinitesimals)
Euler: (17'th century): first formulas of curvature using second derivatives.
Gauss: (18'th century): modern description, higher dimensional versions.

COMPUTING CURVATURE WITH MATHEMATICA

```
\[ \begin{align*}
\mathbf{x}[t_]&=\cos[3\,t]; \\
\mathbf{y}[t_]&=\sin[5\,t]; \\
\mathbf{r}[t_]&=\{\mathbf{x}[t],\mathbf{y}[t]\}; \\
\mathbf{dr}[t_,s]&=\mathbf{r}[s,s], \quad s->t; \\
\mathbf{L}[\{a_,b_\}]&=\sqrt{a^2+b^2}; \\
\mathbf{T}[t_]&=\mathbf{dr}[t]/\mathbf{L}[\mathbf{dr}[t]]; \\
\mathbf{dT}[t_,s]&=\mathbf{T}[s,s], \quad s->t; \\
\kappaappa[t_,s]&=\mathbf{L}[\mathbf{dT}[t]]/\mathbf{L}[\mathbf{dr}[t]]
\end{align*} \]
```