FUNCTIONS, DOMAIN AND RANGE. We deal with functions \( f(x, y) \) of two variables defined on a domain \( D \). The domain is usually the entire plane like for \( f(x, y) = x^2 + \sin(xy) \). But there are cases like in \( f(x, y) = 1/\sqrt{1-(x^2+y^2)} \), where the domain is a subset of the plane. The range is the set of possible values of \( f \).

LEVEL CURVES
2D: If \( f(x, y) \) is a function of two variables, then \( f(x, y) = c = \text{const} \) is a curve or a collection of curves in the plane. It is called contour curve or level curve. For example, \( f(x, y) = 4x^2 + 3y^2 = 1 \) is an ellipse. Level curves allow to visualize functions of two variables \( f(x, y) \).

LEVEL SURFACES. We will later see also 3D analogues: if \( f(x, y, z) \) is a function of three variables and \( c \) is a constant then \( f(x, y, z) = c \) is a surface in space. It is called a contour surface or a level surface. For example if \( f(x, y, z) = 4x^2 + 3y^2 + z^2 \) then the contour surfaces are ellipsoids.

EXAMPLE. Let \( f(x, y) = x^2 - y^2 \). The set \( x^2 - y^2 = 0 \) is the union of the sets \( x = y \) and \( x = -y \). The set \( x^2 - y^2 = 1 \) consists of two hyperbola with with their tips at \((-1, 0)\) and \((1, 0)\). The set \( x^2 - y^2 = -1 \) consists of two hyperbola with their tips at \((0, \pm 1)\).

EXAMPLE. Let \( f(x, y, z) = x^2 + y^2 - z^2 \). \( f(x, y, z) = 0, f(x, y, z) = 1, f(x, y, z) = -1 \). The set \( x^2 + y^2 - z^2 = 0 \) is a cone rotational symmetric around the \( z \)-axis. The set \( x^2 + y^2 - z^2 = 1 \) is a one-sheeted hyperboloid, the set \( x^2 + y^2 - z^2 = -1 \) is a two-sheeted hyperboloid. (To see that it is two-sheeted note that the intersection with \( z = c \) is empty for \(-1 < z < 1\).)

CONTOUR MAP. Drawing several contour curves \( \{ f(x, y) = c \} \) or several contour surfaces \( \{ f(x, y, z) = c \} \) produces a contour map.

The example shows the graph of the function \( f(x, y) = \sin(xy) \). We draw the contour map of \( f \): The curve \( \sin(xy) = c \) is \( xy = C \), where \( C = \arcsin(c) \) is a constant. The curves \( y = C/x \) are hyperbolas except for \( C = 0 \), where \( y = 0 \) is a line. Also the line \( x = 0 \) is a contour curve. The contour map is a family of hyperbolas and the coordinate axis.

TOPOGRAPHY. Topographical maps often show the curves of equal height. With the contour curves as information, it is usually already possible to get a good picture of the situation.
Example. \( f(x, y) = 1 - 2x^2 - y^2 \). The contour curves \( f(x, y) = 1 - 2x^2 + y^2 = c \) are the ellipses \( 2x^2 + y^2 = 1 - c \) for \( c < 1 \).

**SPECIAL LINES.** Level curves are encountered every day:

<table>
<thead>
<tr>
<th>Isobar:</th>
<th>pressure</th>
<th>Isotherm:</th>
<th>temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isocline:</td>
<td>direction</td>
<td>Isoheight:</td>
<td>height</td>
</tr>
</tbody>
</table>

For example, the isobars to the right show the lines of constant temperature in the north east of the US.

A SADDLE. \( f(x, y) = (x^2 - y^2)e^{-x^2-y^2} \). We can here no more find explicit formulas for the contour curves \( (x^2 - y^2)e^{-x^2-y^2} = c \).

Let's try our best:

- \( f(x, y) = 0 \) means \( x^2 - y^2 = 0 \) so that \( x = y, x = -y \) are contour curves.
- On \( y = ax \) the function is \( g(x) = (1 - a^2)x^2e^{-(1+a^2)x^2} \).
- Because \( f(x, y) = f(-x, y) = f(x, -y) \), the function is symmetric with respect to reflections at the \( x \) and \( y \) axis.

A SOMBRERO. The surface \( z = f(x, y) = \sin(\sqrt{x^2 + y^2}) \) has circles as contour lines.

**ABOUT CONTINUITY.** In reality, one sometimes has to deal with functions which are not smooth or not continuous: For example, when plotting the temperature of water in relation to pressure and volume, one experiences *phase transitions*, an other example are water waves breaking in the ocean. Mathematicians have also tried to explain "catastrophic" events mathematically with a theory called "catastrophe theory". Discontinuous things are useful (for example in switches), or not so useful (for example, if something breaks).

**DEFINITION.** A function \( f(x, y) \) is continuous at \((a, b)\) if \( f(a, b) \) is finite and \( \lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) \). The later means that that along any curve \( \tilde{r}(t) \) with \( r(0) = (a, b) \), we have \( \lim_{t\to0} f(\tilde{r}(t)) = f(a, b) \).

Continuity for functions of more variables is defined in the same way.

**EXAMPLE.** \( f(x, y) = (xy)/(x^2 + y^2) \). Because
\[
\lim_{(x,y)\to(0,0)} f(x, x) = \lim_{x\to0} x^2/(2x^2) = 1/2 \quad \text{and} \quad \\
\lim_{(x,0)\to(0,0)} f(0, x) = \lim_{(x,0)\to(0,0)} 0 = 0.
\]
The function is not continuous.

**EXAMPLE.** \( f(x, y) = (x^2y)/(x^2 + y^2) \). In polar coordinates this is \( f(r, \theta) = r^3 \cos^2(\theta) \sin(\theta)/r^2 = r \cos^2(\theta) \sin(\theta) \). We see that \( f(r, \theta) \to 0 \) uniformly if \( r \to 0 \). The function is continuous.