

Rigid local systems on \mathbb{G}_m with finite monodromy

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Katz has constructed some rigid local systems of rank 7 on \mathbb{G}_m in finite characteristics, whose monodromy is a finite subgroup G of the complex Lie group M of type G_2 [K, Thm 9.1]. Among the five finite subgroups G of M which appear in his work, four occur in their natural characteristic:

$SL_2(8)$ in char 2

$PU_3(3)$ in char 3

$PGL_2(7)$ in char 7

$PSL_2(13)$ in char 13.

In this paper, we give another construction of these four local systems on \mathbb{G}_m , and find similar local systems with finite monodromy in other complex Lie groups. Some examples of finite groups which occur in exceptional complex Lie groups M , analogous to the ones above are

$$G = PSL_2(27) \quad \text{in char 3} \quad M = F_4$$

$$G = PSU_3(8) \quad \text{in char 2} \quad M = E_7$$

$$G = PGL_2(31) \quad \text{in char 31} \quad M = E_8$$

$$G = PSL_2(61) \quad \text{in char 61} \quad M = E_8$$

There are interesting families in the classical groups. For example, when $q = p^f > 2$, we have:

$$G = PU_3(q) \quad \text{in char } p \quad M = Sp_{2n} \quad 2n = q(q-1).$$

Our method of construction is as follows. Let k be the finite field with q elements. We first consider the action of a finite group $G = G(q)$ of the type

$$G = PGL_2(q) = A_1(q)$$

$$G = PU_3(q_0) = {}^2A_2(q) \quad q = q_0^2$$

$$G = Sz(q) = {}^2B_2(q) \quad q = 2q_0^2$$

$$G = R(q) = {}^2G_2(q) \quad q = 3q_0^2$$

on the corresponding Deligne-Lusztig curve Y over k . (We note that the algebraic group $PU_3(q_0)$ is defined over q_0 , but is naturally a subgroup of $A_2(q) = PGL_3(q)$ with $q = q_0^2$.) This extends to an action of G on the complete curve $X = Y \cup S$, where S denotes the finite set of points in the completion. Using results of Lusztig [L], we show the quotient map $X \rightarrow X/G$ is ramified over precisely 2 points of the base, which has genus 0. Identifying X/G with \mathbb{P}^1 , and the two points of ramification with $\{0, \infty\}$, we obtain a surjective homomorphism

$$f : \pi_1(\mathbb{G}_m/k) \rightarrow G,$$

where $\mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$.

We then study embeddings

$$g : G \rightarrow M$$

where M is a simple, complex Lie group. Let Ad be the adjoint representation of M on its Lie algebra. The composite homomorphism

$$\pi_1(\mathbb{G}_m/k) \xrightarrow{f} G \xrightarrow{g} M \xrightarrow{Ad} GL(\text{Lie}(M))$$

gives a complex representation of the Galois group of the global field $k(\mathbb{G}_m) = k(t)$, which is unramified outside $t = 0, \infty$. We say the homomorphism g is *rigid* if the Artin L -function $L(\text{Ad}, s)$ is equal to the constant function 1.

The embedding g is rigid precisely when the ℓ -adic sheaf on \mathbb{P}^1 corresponding to the middle extension of the adjoint representation has vanishing cohomology. This is equivalent to the statement that the principal M -bundle on \mathbb{G}_m has no deformations, preserving the singularities at $t = 0$ and $t = \infty$.

Using results of Serre [S] and Greiss-Ryba [G-Ry], we construct rigid embeddings for $G = PGL_2(q)$ and $G = PU_3(q_0)$, as well as for their normal subgroups

$$G_0 = PSL_2(q) \quad q \equiv 1 \pmod{2}.$$

$$G_0 = PSU_3(q_0) \quad q_0 \equiv 2 \pmod{3}.$$

These give rise to the rigid local systems on \mathbb{G}_m with finite monodromy, described above.

As a bonus, we obtain simple wild parameters (cf. [G-Re, §5.6]) for the local field $k((1/t))$, by restricting our local systems to the decomposition group at $t = \infty$. These parameters are representations of the local Galois group into M , which have no inertial invariants on the adjoint representation and have Swan conductor equal to the rank of M .

1. Deligne-Lusztig curves

Let $d = 2, 3, 4, 6$. For each d , and certain prime powers q , we consider the Deligne-Lusztig curve $Y = Y(w)$ over the field k with q elements. the restrictions on q are

$$\begin{aligned} d = 2 & \quad q = \text{arbitrary} \\ 3 & \quad q = q_0^2 \\ 4 & \quad q = 2q_0^2 \\ 6 & \quad q = 3q_0^2 \end{aligned}$$

The curve Y is affine and non-singular, and admits an action of the finite adjoint group $G = G(q)$ over k . In ATLAS notation, G is given by

$$\begin{aligned} d = 2 & \quad PGL_2(q) = A_1(q) \\ 3 & \quad PU_3(q_0) = {}^2A_2(q) \\ 4 & \quad Sz(q) = {}^2B_2(q) \\ 6 & \quad R(q) = {}^2G_2(q) \end{aligned}$$

The order of G is given by the formula

$$\#G = q^{d/2}(q^{d/2+1})(q-1).$$

Let \bar{k} be an algebraic closure of k and $F : G(\bar{k}) \rightarrow G(\bar{k})$ the (geometric) Frobenius map with fixed points G . By definition, Y is the variety of all Borel subgroups of $G(\bar{k})$ such that B and FB are in relative position w , where w is the Coxeter class in the Weyl group. The integer d is the order of the

Coxeter class w . (In [L] the affine curve Y is denoted X_f , and the integer d is denoted h_0 .) Since G acts on the function field $k(Y)$, the action on Y extends to an action on the complete non-singular model $X = \overline{Y} = Y \cup S$, where S denotes the finite set of points in the completion.

There are two maximal subgroups of G that will play an important role in the study of the G action on X . The first is the Borel subgroup $B = B(q)$, of order $q^{d/2}(q-1)$. This fixes a point in a 2-transitive permutation representation, which we will see occurs on the points S of the completion. The p -Sylow subgroup U of B is normal, and the quotient B/U is cyclic. The second maximal subgroup is the normalizer $N(T)$ of a Coxeter torus $T = T(q)$. This is the semi-direct product of the cyclic group T , of order

$$\begin{array}{ll} d = 2 & q + 1 \\ & 3 \quad q - q_0 + 1 \\ & 4 \quad q - 2q_0 + 1 \\ & 6 \quad q - 3q_0 + 1 \end{array}$$

dividing $(q^{d/2} + 1)$, with a Coxeter element of order d in G : $N(T) = T.d$.

The adjoint group $G = G(q)$ has a normal subgroup G_0 , which is the image of k -rational points of the simply-connected cover. This has index > 1 in the following cases

$$\begin{array}{lll} d = 2 & PGL_2(q) \triangleright_2 PSL_2(q) & q \equiv 1 \pmod{2} \\ d = 3 & PU_3(q_0) \triangleright_3 PSU_3(q_0) & q_0 \equiv 2 \pmod{3} \end{array}$$

Both B and T have a non-trivial image in the quotient group G/G_0 in these cases, so the kernel defines maximal subgroups B_0 and $N(T_0) = T_0.d$ in G_0 .

The action of G on the affine curve Y is completely described in [L]. We summarize these results in Theorem 1 below. For $m \geq 1$, let k_m be the unique extension of k of degree m in \bar{k} .

Theorem 1 (Lusztig)

- a) *The set $Y(k_m)$ is empty, for $1 \leq m < d$.*
- b) *G acts transitively on $Y(k_d)$ with stabilizer T .*
- c) *G acts freely on $Y(\bar{k}) - Y(k_d)$, and acts simply-transitively on $Y(k_{d+1})$.*
- d) *The linear action of G on the compactly supported ℓ -adic cohomology groups $H_c^i(Y/\bar{k}, \mathbb{Q}_\ell)$ commutes with the action of F , and the distinct eigenspaces for F give distinct unipotent representations W of G . A unipotent representation occurs in $H_c^i(Y)$ if and only if $\text{Tr}(t|W) = (-1)^i$, for all regular semi-simple elements t in T . If such a representation occurs, its multiplicity in $H_c^i(Y)$ is equal to one.*

One can make part d) more explicit by enumerating the eigenvalues α of F on $H_c^i(Y)$, and the unipotent representation W_α that appears in the α -eigenspace. They are

$$\begin{aligned}
 \alpha = q \quad \text{on } H_c^2(Y) & \quad W = \text{trivial representation} \\
 \alpha = 1 \quad \text{on } H_c^1(Y) & \quad W = \text{Steinberg representation} \\
 & \quad \dim W = q^{d/2}
 \end{aligned}$$

There are $(d - 2)$ further eigenvalues on $H_c^1(Y)$, which correspond to unipotent cuspidal representation

$$\begin{aligned} d &= 3 \\ \alpha &= -q_0 & \dim W &= q_0(q_0 - 1) \end{aligned}$$

$$\begin{aligned} d &= 4 \\ \alpha_{\pm} &= -q_0(1 \pm i) & \dim W_{\pm} &= q_0(q - 1) \end{aligned}$$

$$\begin{aligned} d &= 6 \\ \alpha_{\pm} &= -q_0 \left(\frac{3 \pm \sqrt{-3}}{2} \right) & \dim W_{\pm} &= q_0(q^2 - 1) \end{aligned}$$

$$\beta_{\pm} = -q_0(\pm\sqrt{-3}) \quad \dim W_{\pm} = q_0(q - 1)(q + 3q_0 + 1)/2$$

The two unipotent cuspidal representations W of the Ree groups with

$$\dim W = q_0(q - 1)(q - 3q_0 + 1)/2$$

do **not** appear in the cohomology of Y , as they restrict to the regular representation of T .

From Lusztig's fundamental results, we deduce the following facts on the complete curve $X = \bar{Y} = Y \cup S$.

Corollary

- a) G acts transitively on $X(k) = S(k) = S(\bar{k})$, with stabilizer B .
- b) The covering $X \rightarrow X/G = Z$ is ramified over two rational points of the base, which has genus 0.
- c) We may fix an isomorphism $Z \rightarrow \mathbb{P}^1$ over k so that the resulting cover $\pi : X \rightarrow \mathbb{P}^1$ satisfies

$$\pi(X(k)) = \infty$$

$$\pi(X(k_d)) - X(k) = 0$$

$$\pi(X(k_{d+1}) - X(k)) = 1$$

d) *The decomposition group D , inertia group I , and wild inertia group P at the points $\{\infty, 0, 1\}$ are*

$$D_\infty = I_\infty = B \triangleright P_\infty = U \quad I_\infty/P_\infty = (q-1)$$

$$D_0 = N(T) \triangleright I_0 = T \quad P_0 = 1 \quad D_0/I_0 = (d)$$

$$D_1 = (d+1) \quad I_1 = 1.$$

Proof. Let W_0 be the representation of G on the functions on S with values in \mathbb{Q}_ℓ and with sum equal to 0. Then W_0 appears in $H_c^1(Y)$, and the eigenvalues of F on W_0 are roots of unity. The only such eigenvalue is $\alpha = 1$, so $S(k) = S(\bar{k})$. This eigenspace affords the Steinberg representation of G , which is the functions on G/B with sum equal to 0. Hence G acts transitively on $S(k) = X(k)$ with stabilizer B .

Since G fixes no non-zero vector in $H^1(X) = \bigoplus_{i=1}^{d-2} W_i$ the quotient curve $Z = X/G$ has genus 0. The decomposition groups at $\{\infty, 0, 1\}$ and their inertia subgroups follows from the structure of the G -orbits on $X(\bar{k})$, and their fields of rationality. We note that the element of order $(d+1)$ generating D_1 is a torsion class of Kac type $(2; 1, 1)$ in $G(q)$, just as the element of d normalizing T is of Coxeter type $(1; 1, 1)$.

Let $K = k(\mathbb{P}^1) = k(t)$, and let $K' = k(X)$ be the Galois extension of K corresponding to the covering $X \rightarrow \mathbb{P}^1$. Let V be a complex representation of $G = \text{Gal}(K'/K)$, and for each place v of K , let F_v be an geometric Frobenius

element in the quotient G_v/I_v . We define the Artin L -function of V by an Euler product over the places v of K :

$$L(V, x) = \prod_v \det(1 - F_v x^{\deg v} | V^{I_v})^{-1}.$$

This is a rational function of x , and is a polynomial in x when $V^G = 0$. We define the Artin conductor $f_v(V)$ and the Swan conductor $sw_v(V)$ as in [S2, Ch VI]. Then

$$f_0(V) = \dim(V/V^T)$$

$$f_\infty(V) = \dim(V/V^B) + sw_\infty(V).$$

Since the degree of $L(V, x)$ is given by [W, Appendix V]

$$\deg L(V, x) = f_0(V) + f_\infty(V) - 2 \dim V$$

we find the formula

$$sw_\infty(V) = \deg L(V, x) + \dim V^B + \dim V^T.$$

Lusztig's calculation of the cohomology of Y gives us a determination of the Artin L -functions of the irreducible representations W of G , as we have the factorization

$$\prod_{\substack{\text{irred} \\ W}} L(W, x)^{\dim W} = \zeta_{K'}(x) = \frac{\det(1 - Fx | H^1(X))}{(1-x)(1-qx)}.$$

Corollary. *If W is an irreducible representation of G , then $L(W, x)$ has degree = -2, 0, or 1. More precisely*

a) if W is the trivial representation

$$L(W, x) = \frac{1}{(1-x)(1-qx)}$$

b) if W is a unipotent cuspidal representation, which occurs as the α -eigenspace for F in $H^1(X)$, then

$$L(W, x) = (1 - \alpha x)$$

c) for all other irreducible representations W , $L(W, x) = 1$.

Corollary. a) If W is the trivial representation of G , or the quadratic character of $PGL_2(q)$ when $q \equiv 1 \pmod{2}$, or one of the cubic characters of $PU_3(q_0)$ when $q_0 \equiv 2 \pmod{3}$ then $sw_\infty(W) = 0$.

b) If W is an irreducible representation of $PGL_2(q)$ and $\dim W > 1$, then $sw_\infty(W) = 1$.

c) If W is the unipotent cuspidal representation of $PU_3(q_0)$, of dimension $q_0^2 - q_0$, then $sw_\infty(W) = 1$.

Proof. a) These are precisely the irreducible representations which are tamely ramified. For all other irreducible representations, $sw_\infty(W) \geq 1$.

b) For these representations, $L(W, x) = 1$ has degree 0, so $sw_\infty(W) = \dim W^B + \dim W^T$. For the Steinberg representation $W = St$ of dimension q , we have $\dim W^B = 1$ and $\dim W^T = 0$. For all other irreducibles (of dimensions $q - 1, q, q + 1$) we have $\dim W^B = 0$ and $\dim W^T = 1$.

c) In this case, $L(W, x) = (1 + q_0x)$ has degree 1. We find $\dim W^B =$

$\dim W^T = 0$, as W is cuspidal and the restriction of W to T is the regular representation minus the trivial character. Hence $sw_\infty(W) = 1$.

The Swan conductors of all other irreducible representations W of G grow with the size of q . For example, the Steinberg representation St of $G(q)$ has

$$\begin{aligned} sw_\infty(St) &= \dim St^B + \dim St^T \\ &= 1 + \dim St^T \\ &= (q^{d/2} + 1)/\#T \\ &= \begin{cases} 1 & d = 2 \\ q_0 + 1 & d = 3 \\ q + 2q_0 + 1 & d = 4 \\ q^2 + 3qq_0 + 2q + 3q_0 + 1 & d = 6 \end{cases} \end{aligned}$$

The two unipotent cuspidal representations W of the Ree group ${}^2G_2(q)$ which do **not** appear in the cohomology of X have

$$sw_\infty(W) = \dim W^T = q_0 \left(\frac{q-1}{2} \right).$$

We now make some observations about the subfields of $K' = k(X)$ fixed by certain subgroups of the Galois group G . In the cases where we have a normal subgroup $G \triangleright G_0$:

$$PGL_2(q) \triangleright_2 PSL_2(q) \quad q \equiv 1 \pmod{2}.$$

$$PU_3(q_0) \triangleright_3 PSU_3(q_0) \quad q_0 \equiv 2 \pmod{3}.$$

The fixed field K_0 is tamely ramified, of degree $d = 2$ or 3 , over $K = k(t)$. Since it is also split at the place $t = 1$, we find

$$K_0 = k(\sqrt[d]{t}).$$

Since the fixed field K_0 of G_0 also has genus 0, we obtain d homomorphisms up to conjugacy

$$f_0 : \pi_1(\mathbb{G}_m/k) \rightarrow G_0.$$

These are all conjugate under the other action of G . The different classes of f_0 correspond to fixing a d^{th} root of t in K_0 , or equivalently, to fixing a place above the place $t = 1$ in K .

Now consider the subfield M of K' fixed by the subgroup B of G :

$$\begin{array}{ccc} & & K' = k(X) \\ & B/ & \\ & M & \\ & \searrow & \\ & & K = k(t) \end{array}$$

This has degree $(q^{d/2} + 1)$ over K . Since $H^1(X)^B = 0$, it also has genus 0.

The zeta function of M factors

$$\zeta_M(x) = \zeta_K(x)L(St, x)$$

and the discriminant divisor $D(M/K)$ of M/K is given by the conductor of the Steinberg representation:

$$D(M/K) = f_\infty(St).(\infty) + f_0(St)(0).$$

We have

$$\begin{aligned}
f_\infty(St) &= \dim(St/St^B) + sw_\infty(St) \\
&= (q^{d/2} - 1) + sw_\infty(St) \\
&= \begin{cases} q & d = 2 \\ q_0^3 + q_0 & d = 3 \\ q^2 + q + 2q_0 & d = 4 \\ q^3 + q^2 + 3qq_0 + 2q + 3q_0 & d = 6 \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
f_0(St) &= \dim(St/St^T) \\
&= \begin{cases} q & d = 2 \\ q_0^3 - q_0 & d = 3 \\ q^2 - q - 2q_0 & d = 4 \\ q^3 - q^2 - 3qq_0 - 2q - 3q_0 & d = 6 \end{cases}
\end{aligned}$$

The prime P_∞ factors in M as

$$P_\infty = R \cdot (R')^{q^{d/2}} \quad \text{with } \deg R = \deg R' = 1$$

The prime P_0 factors in M as

$$P_0 = \prod_{i=1}^n (R_i)^{\#T} \quad \deg R_i = 1$$

where $n = (q^{d/2} + 1)/\#T$. All other primes are unramified.

In the case where $G = PGL_2(q)$, the curve X is the projective line over $k = \mathbb{F}_q$ and the action is by fractional linear transformations. Here, specific

polynomials of degree $(q+1)$ over $K = k(t)$ are known, with root field M and splitting field $K' = k(X)$. Can one find such equations, of degree $(q^{d/2} + 1)$ over K , in general?

2. Rigid local systems

In the previous section, we defined a surjective homomorphism

$$f : \pi_1(\mathbb{G}_m/k) \rightarrow G$$

where k is a finite field with q elements and G is the finite group acting on a Deligne-Lusztig curve X over k (or a normal subgroup G_0 of G).

In this section, we will consider the local systems on \mathbb{G}_m/k that we obtain by composing f with a homomorphism

$$g : G \rightarrow M,$$

where M is a connected, simple complex Lie group. Consider the adjoint representation of M on its Lie algebra. The composite homomorphism

$$\pi_1(\mathbb{G}_m/k) \xrightarrow{f} G \xrightarrow{g} M \longrightarrow GL(\text{Lie}(M))$$

defines a complex orthogonal representation Ad of the Galois group of the global field $K = k(t)$, which is unramified outside $t = 0$ and $t = \infty$. We say the homomorphism g is *rigid* if the Artin- L -function $L(\text{Ad}, s)$ is identically equal to 1.

If we identify the complex numbers with an algebraic closure of \mathbb{Q}_ℓ , the representation Ad of $\pi_1(\mathbb{G}_m/k)$ gives a lisse λ -adic sheaf \mathcal{F} of rank equal to $\dim(M)$ on \mathbb{G}_m . Let $j : \mathbb{G}_m \hookrightarrow \mathbb{P}^1$ be the inclusion. Then the middle extension $j_*\mathcal{F}$, whose first cohomology gives the image of cohomology with compact supports in the cohomology of \mathcal{F} over \mathbb{G}_m , is pure of weight 0 on \mathbb{P}^1 . By Grothendieck's cohomological formula for $L(\text{Ad}, s) = L(j_*\mathcal{F}, s)$, the

adjoint L -function is constant if and only if the ℓ -adic cohomology groups of $j_*\mathcal{F}$ all vanish: $H^i(\mathbb{P}^1, j_*\mathcal{F}) = 0$ for $i = 0, 1, 2$. This implies that the M -local system on \mathbb{G}_m has no deformations preserving the singularities at $t = 0, \infty$. Therefore rigid maps g give rise to rigid M -local systems on \mathbb{G}_m .

First, from the determination of $L(V, s)$ for irreducible representations V of G in §1, we obtain

Proposition. *The homomorphism $g : G \rightarrow M$ is rigid if and only if*

- 1) $\text{Ad}^G = 0$
- 2) *the restriction of Ad to G does not contain any of the $(d-2)$ unipotent cuspidal representations which appear in $H^1(X)$.*

When $G = PGL_2(q)$ or $PSL_2(q)$, we have $d = 2$ and there are no unipotent cuspidal representations. Hence the only condition for rigidity is that $\text{Ad}^G = 0$, or equivalently that the centralizer of the image of G in M is finite. We now describe some rigid embeddings in this case, by appealing to the work of Serre [S] and Greiss-Ryba[G-Ry] on finite subgroups of Lie groups.

Proposition. 1) *Assume that $G = PGL_2(q)$ and that M is of adjoint type, with Coxeter number $h = q - 1$. Then there are rigid homomorphism*

$$g : G \rightarrow M.$$

- 2) *Assume $q \equiv 1 \pmod{2}$, that $G = PSL_2(q)$, and that M is of adjoint*

type, with Coxeter number $h = \frac{q-1}{2}$. Then there are rigid homomorphisms

$$g : G \rightarrow M.$$

Proof. If q is a prime number p , Serre [S] has constructed are principal embeddings:

$$g : PGL_2(p) \rightarrow M \quad p = h + 1,$$

$$g : PSL_2(p) \rightarrow M \quad p = 2h + 1.$$

These satisfy $\text{Ad}^G = 0$. The only case where $q \neq p$ in the exceptional series is for M of type F_4 and E_6 , where $h = 12$ and $2h + 1 = 25 = 5^2$. In this case, a rigid embedding

$$g : PSL_2(25) \rightarrow M$$

has been constructed by [G-Ry].

Hence, we may assume that M is an adjoint group of classical type A, B, C, D . We tabulate M , its Coxeter number h , and a central covering $\widetilde{M} \rightarrow M$ which is a connected classical group: $\widetilde{M} \subset GL(V)$.

M	h	\widetilde{M}
PGL_n	n	GL_n
PSp_{2n}	$2n$	Sp_{2n}
SO_{2n+1}	$2n$	SO_{2n+1}
PSO_{2n+2}	$2n$	SO_{2n+2}

We will construct a rigid homomorphism $g : G \rightarrow M$ by constructing a complex representation V of G (or a cover $2.G$) with the appropriate dimension and self-duality.

Assume that $q = h + 1$. If $M = PGL_n$, we let V be an irreducible representation of $PGL_2(q)$ in the discrete series. These representations are all of dimension $q - 1$, so give

$$G \rightarrow GL_{q-1} \rightarrow PGL_{q-1} = PGL_n = M$$

. Since $Ad = V \otimes V^* - 1$, we have $Ad^G = 0$.

In the other cases, $h = 2n$ is even and $q = h + 1 = 2n + 1$ is odd. For $M = SO_{2n+1}$, we let S be the Steinberg representation of $G = PGL_2(q)$, which is orthogonal of dimension $q = 2n + 1$. The determinant $\epsilon = \det(S)$ is the nontrivial quadratic character of G , so the representation $V = S \otimes \epsilon$ gives a homomorphism

$$G \rightarrow SO_q = SO_{2n+1} = M.$$

In this case, $Ad = \wedge^2 V$, so the irreducibility forces $Ad^G = 0$. We could also take $V = W \oplus \epsilon$, where W is an (orthogonal) discrete series representation of dimension $q - 1 = 2n$ and determinant ϵ . Then $Ad = \wedge^2 W \oplus (W \otimes \epsilon)$ also has no G -invariants.

For $M = PSO_{2n+2}$ we can take either $V = St \oplus \epsilon$ or $V = W \oplus \epsilon \oplus 1$. Both are orthogonal representations of dimension $2n + 2$ and determinant 1, so give

$$g : G \rightarrow SO_{2n+2} \rightarrow PSO_{2n+2} = M$$

with $Ad^G = (\wedge^2 V)^G = 0$.

Finally, for $M = PSp_{2n}$ we take V an irreducible representation of dimension $2n = q - 1$ of a double cover $\tilde{G} = 2.G = 2.PSL_2(q).2$ which is isoclinic

to the group $GL_2(q)/\left\{\begin{pmatrix} a^2 & \\ & a^2 \end{pmatrix}\right\}$. The representation V has a real character, but is obstructed. It gives a symplectic representation:

$$\begin{array}{ccc} \tilde{G} & \longrightarrow & Sp_{2n} \\ \downarrow & & \downarrow \\ G & \longrightarrow & PSp_{2n} \end{array}$$

The case when $q = 2h + 1$ is similar. Here we use the half-discrete series of dimension $\frac{q-1}{2}$ for $G = PSL_2(q)$ in the case where $\tilde{M} = Sp$, and the half-principal series of dimension $\frac{q+1}{2}$ for $G = PSL_2(q)$ in the cases where $\tilde{M} = SO$. We leave the details to the reader.

Note. In contrast to the case for $G = PGL_2(q)$ when $q = h + 1$, where there are many rigid embeddings, when $G = PSL_2(q)$ and $q = 2h + 1$ we can only construct *two* rigid embeddings, into M , which are equivalent under the outer action of $PGL_2(q)$.

Corollary. *For the maps $g : G \rightarrow M$ described above, when $q = h + 1$ and $q = 2h + 1$, the restriction of $g \circ f$ to the decomposition group D_∞ in $\text{Gal}(K^s/K)$ is a simple wild parameter for the split, simply-connected group over the local field $K_\infty = k((1/t))$ which is dual to M .*

Proof. We must check the two conditions [G-Re,§5.6]:

$$\text{Ad}^{D_\infty} = 0$$

$$sw_\infty(\text{Ad}) = \ell = \text{rank}(M).$$

The first follows from the fact that the Steinberg representation St does not appear in Ad . In the exceptional cases, this follows from the work of Serre [S] and Greiss-Ryba [G-Ry], and in the classical cases from a determination of the decomposition of S^2V and \wedge^2V . We then have the formula

$$sw_\infty(\text{Ad}) = \dim \text{Ad}^T.$$

But T acts regularly on the discrete series representations, and this allows us to calculate $\dim \text{Ad}^T = \ell$.

We now consider rigid embeddings of the groups $G = PU_3(q_0)$ and $G = PSU_3(q_0)$. The group $G = PU_3(q_0)$ has a unique unipotent cuspidal representation V of dimension $2n = q_0(q_0 - 1)$. This representation has rational character, but is obstructed by the rational quaternion algebra ramified at ∞ and p (cf. [G]). It therefore gives a homomorphism

$$g : G \rightarrow M = Sp(V) \hookrightarrow SL(V).$$

Proposition. 1) *The restriction of $g \circ f$ to the decomposition group D_∞ in $\text{Gal}(K^s/K)$ is a simple wild parameter for both groups SO_{2n+1} and PGL_{2n} over $K_\infty = \mathbb{F}_q((1/t))$.*

2) *The homomorphism g is rigid.*

Proof. The restriction of V to the subgroup $D_\infty = B = q_0^{1+2}(q_0^2 - 1)$ is irreducible, so there are no B -invariants in either S^2V or in $(\wedge^2V)_0$. We have

also seen that $sw_\infty(V) = 1$ and that T acts as the regular representation on $V \oplus 1$.

We now show that

$$sw_\infty(S^2V) = \dim(S^2V)^T = n$$

$$sw_\infty(\wedge^2V_0) = \dim(\wedge^2V_0)^T = n - 1$$

which together imply that

$$L(S^2V, t) = L(\wedge^2V_0, t) = 1.$$

The T -invariants in S^2V and in $(\wedge^2V)_0$ follow immediately from the determination of the characters of T on V . To calculate the Swan conductor, we observe that the group $B = q_0^{1+2}(q_0^2 - 1)$ has irreducible representations of dimensions $q_0^2 - q_0$, $q_0^2 - 1$, and 1, with Swan conductors 1, 1, and 0, respectively. The representations of dimension $q_0^2 - q_0$ have nontrivial central characters on q_0^{1+2} , and the representation of dimension $q_0^2 - 1$ has trivial central character, but is nontrivial on q_0^{1+2} . From this, the Swan conductors of S^2V and $(\wedge^2V)_0$ are easily determined.

Since the adjoint representations are

$$\text{Ad} = S^2V \quad \text{for } M = Sp(V)$$

$$\text{Ad} = S^2V + (\wedge^2V)_0 \quad \text{for } M = SL(V)$$

We see that $g \circ f$ is a simple wild parameter for the groups SO_{2n+1} and PGL_2 over K_∞ , and that the homomorphism g is rigid. This completes the proof.

Note. When $q_0 = 8$, we have $2n = q_0^2 - q_0 = 56$. In this case, the subgroup $G_0 = PSU_3(8)$ of index 3 in G stabilizes a symmetric quartic form on V , and we obtain an embedding [G-Ry]:

$$g : PSU_3(8) \rightarrow E_7 \subset Sp(V)$$

which is also rigid. The restriction of

$$\pi_1(\mathbb{G}_m/\mathbb{F}_{64}) \xrightarrow{f} PSU_3(8) \xrightarrow{g} E_7$$

to $D_\infty = 8^{1+2}.21 = 2^{3+6}.21$ is a simple wild parameter for E_7 over $\mathbb{F}_{64}((1/t))$.

We have:

$$\begin{aligned} sw_\infty(V) &= 3 & \deg L(V, t) &= 1 \\ sw_\infty(\text{Ad}) &= 7 & \deg L(\text{Ad}, t) &= 0. \end{aligned}$$

We have now constructed all of Katz's local systems for G_2 , with the exception of the group $SL_2(8)$ in char 2. The example of $SL_2(8)$ fits into a series of embeddings [S, pg. 422]

$$\begin{array}{lll} SL_2(4) & \text{in} & SO_3 = A_1^{ad} \\ SL_2(8) & \text{in} & G_2 \\ SL_2(16) & \text{in} & D_8^{ad} \\ SL_2(32) & \text{in} & E_8 \\ PSL_2(9) & \text{in} & A_2^{ad} \\ PSL_2(27) & \text{in} & F_4 \end{array}$$

where the image of the Sylow p -subgroup is not contained in a maximal torus. We leave it to the reader to check that these embeddings are all rigid (the first four in char 2 and the last two in char 3). In each case, the restriction to D_∞ is a simple wild parameter.

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