

# Winning sets, quasiconformal maps and Diophantine approximation

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## Abstract

This paper describes a class of *strong winning sets* in  $\mathbb{R}^n$ , defined using a variant of Schmidt's game. This class includes many Diophantine sets of measure zero and first category, and has good behavior under countable intersections. Most notably, it is invariant under quasiconformal maps, while the class of classical winning sets is not.

## Contents

1	Introduction . . . . .	1
2	Transport of strategies . . . . .	4
3	Diophantine sets . . . . .	6
4	Counterexamples . . . . .	11

## 1 Introduction

**Winning sets.** Suppose two players,  $A$  and  $B$ , take turns choosing a nested sequence of closed Euclidean balls

$$B_1 \supset A_1 \supset B_2 \supset A_2 \supset B_3 \cdots$$

in  $\mathbb{R}^n$  whose diameters satisfy, for fixed  $0 < \alpha, \beta < 1$ ,

$$|A_i| = \alpha|B_i| \quad \text{and} \quad |B_{i+1}| = \beta|A_i|. \quad (1.1)$$

Following Schmidt [Sch], we say  $E \subset \mathbb{R}^n$  is an  $(\alpha, \beta)$ -*winning set* if player  $A$  has a strategy which guarantees that  $\bigcap A_i$  meets  $E$ . We say

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$E$  is an  $\alpha$ -winning set if it is  $(\alpha, \beta)$ -winning for all  $0 < \beta < 1$ , and a winning set if it is  $\alpha$ -winning for some  $\alpha > 0$ .

Winning sets have many useful properties; for example:

1. Any winning set in  $\mathbb{R}^n$  has Hausdorff dimension  $n$ .
2. A countable intersection of  $\alpha$ -winning sets is  $\alpha$ -winning.
3. Winning sets are preserved by bi-Lipschitz homeomorphisms of  $\mathbb{R}^n$ .

See [Sch] and [Dani3, Prop 5.3]. However, as we will see in §4:

**Theorem 1.1** *Winning sets are generally not preserved by quasisymmetric maps.*

Here a map  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $k$ -quasisymmetric if for all balls  $B(x, r) \subset \mathbb{R}^n$  and  $r > 0$ , there exist an  $s > 0$  such that

$$B(\phi(x), s) \subset \phi(B(x, r)) \subset B(\phi(x), ks). \quad (1.2)$$

In dimensions  $n \geq 2$ , quasisymmetric maps are the same as quasiconformal maps [Geh2, Thm 7.7].

**Strong winning.** In this note we propose a variant of Schmidt's game, in which (1.1) is replaced by the pair of inequalities:

$$|A_i| \geq \alpha |B_i| \quad \text{and} \quad |B_{i+1}| \geq \beta |A_i|. \quad (1.3)$$

If  $A$  has a winning strategy in the game defined by (1.3), we say  $E$  is an  $(\alpha, \beta)$ -strong winning set. Sets which are  $\alpha$ -strong winning and strong winning are defined analogously.

It is straightforward to verify that:

1. A strong winning set  $E \subset \mathbb{R}^n$  is winning; in particular,  $\text{H. dim}(E) = n$ ; and
2. A countable intersection of  $\alpha$ -strong winning sets is  $\alpha$ -strong winning.

In addition, property (3) for winning sets can be strengthened to:

**Theorem 1.2** *Strong winning sets are preserved by quasisymmetric homeomorphisms  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .*

See §2.

**Diophantine sets.** Now let  $\Gamma \subset \text{Isom}(\mathbb{H}^{n+1})$  be a lattice in the isometry group of the hyperbolic plane, with a cusp at infinity. Then the endpoints of lifts of all bounded geodesic rays in  $\mathbb{H}^{n+1}/\Gamma$  determine a *Diophantine set*

$$D(\Gamma) \subset \mathbb{R}^n \subset \partial\mathbb{H}^{n+1}.$$

For example,

$$D(\mathrm{SL}_2(\mathbb{Z})) = \left\{ x \in \mathbb{R} : \exists C > 0 : \left| x - \frac{p}{q} \right| > \frac{C}{q^2} \text{ for all } p/q \in \mathbb{Q} \right\}$$

consists of the real numbers which are badly approximable by rationals. In general  $D(\Gamma)$  consists of the points in  $\mathbb{R}^n$  which are badly approximable by the cusps of  $\Gamma$  (see §3).

It is known that  $D(\Gamma)$  is a winning set [Dani1] (see [Sch] for the case of  $\mathrm{SL}_2(\mathbb{Z})$ ). Sharpening this result, in §3 we give a geometric proof of:

**Theorem 1.3** *For any lattice  $\Gamma \subset \mathrm{Isom}(\mathbb{H}^{n+1})$  with a cusp at infinity, the Diophantine set  $D(\Gamma)$  is strong winning.*

**Corollary 1.4** *Let  $D = D(\mathrm{SL}_2(\mathbb{Z}))$  be the set of badly approximable numbers. Then for any sequence of  $k$ -quasisymmetric maps  $\phi_i : \mathbb{R} \rightarrow \mathbb{R}$ , the set  $\bigcap \phi_i(D)$  has Hausdorff dimension one (but measure zero).*

**Example: Fuchsian groups.** To motivate Theorem 1.2, consider a homeomorphism  $h : X_1 \rightarrow X_2$  between a pair of finite-volume hyperbolic Riemann surfaces, each with a unique cusp. Their universal covers can be normalized so that  $X_i = \mathbb{H}/\Gamma_i$ ,  $i = 1, 2$  and  $\Gamma_i$  has a cusp at  $z = \infty$ . Then  $h$  can be lifted to a homeomorphism  $\tilde{h} : \mathbb{H} \rightarrow \mathbb{H}$  whose boundary values give a quasisymmetric map

$$\phi : \mathbb{R} \rightarrow \mathbb{R}.$$

The map  $\phi$  depends only on the isotopy class of  $h$ , and is compatible with an isomorphism  $\Gamma_1 \cong \Gamma_2$ .

Now assume  $h$  is not isotopic to an isomorphism. Then  $\phi$  is highly singular: it does not preserve sets of measure zero, and it even sends a set of full measure to a set of Hausdorff dimension  $< 1$  (cf. [Ku], [Mc]). Nevertheless, the bounded geodesic rays in  $X_1$  and  $X_2$  determine a pair of Diophantine sets  $D_1$  and  $D_2$  of Hausdorff dimension one satisfying  $\phi(D_1) = D_2$ . Thus it is natural to ask if  $\mathrm{H. dim} \phi(D_1) = 1$  for all quasisymmetric  $\phi$ . A positive answer is provided by Theorem 1.2.

**Remark: Porous sets.** A set  $E \subset \mathbb{R}^n$  is *porous* if there exists a  $\delta > 0$  such that for all  $x \in \mathbb{R}^n$  and  $r > 0$ ,  $B(x, r)$  contains a ball  $B(y, \delta r)$  disjoint from  $E$ . A countable union of porous sets is  $\sigma$ -porous. These sets enjoy properties parallel to those of strong winning sets; for example, one can readily verify [Vai2]:

1. A  $\sigma$ -porous set in  $\mathbb{R}^n$  has measure zero and is meager in the sense of Baire category.
2. A countable union  $\sigma$ -porous sets is  $\sigma$ -porous.

3. Quasisymmetric mappings preserve  $\sigma$ -porous sets.

We will also see:

**Proposition 1.5** *For any lattice with a cusp at infinity,  $D(\Gamma)$  is  $\sigma$ -porous.*

Thus any Diophantine set  $D(\Gamma)$  is simultaneously large (in the sense of games) and small (in the sense of measure, category and porosity), and both properties are preserved under quasisymmetric maps. (This explains the measure zero remark in Corollary 1.4.)

For additional perspective, in §3 we define a *steering game* in which  $A$  and  $B$  take turns constructing a quasigeodesic ray  $\gamma \subset \mathbb{H}^{n+1}$ . Then player  $A$  can guide the endpoint of  $\gamma$  into  $E \subset \partial\mathbb{H}^{n+1}$  iff  $E$  is a strong winning set.

**Question.** Do the bounded Teichmüller geodesics in  $\mathcal{M}_{g,n}$  determine a strong winning set in  $\mathbb{P}\mathcal{ML}_{g,n}$ ?

**Notes and references.** Winning sets arise naturally in many other settings in dynamics and Diophantine approximation (see e.g. [Dani2], [Ts], [Fi], [KW]); it seems likely that most of these sets are also strong winning.

Schmidt's game is a constrained form of the Banach–Mazur game, which is discussed in [Ox] and [Te]. Another counterexample for Schmidt's game is given in [Fr].

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**Notation.** The expression  $A \asymp B$  means  $0 < A/C < B < CA$  for an implicit constant  $C$ . The closed ball of radius  $r$  about  $x$  is denoted  $B(x, r)$ , and  $B(E, r) = \bigcup_{x \in E} B(x, r)$  for any set  $E$ . The diameter of a set is given by  $|E| = \sup\{d(x, y) : x, y \in E\}$ .

We use  $\mathbb{N}$  to denote the natural numbers  $\{0, 1, 2, \dots\}$ . The decomposition of a real number into its integral and fractional parts is given by  $x = \lfloor x \rfloor + (x \bmod 1)$ ; here  $\lfloor x \rfloor \in \mathbb{Z}$  and  $x \bmod 1 \in [0, 1)$ .

## 2 Transport of strategies

In this section we prove the following uniform version of Theorem 1.2.

**Theorem 2.1** *The image of an  $\alpha$ -strong winning set  $E \subset \mathbb{R}^n$  under a  $k$ -quasisymmetric map  $\phi$  is  $\alpha'$ -strong winning, where  $\alpha'$  depends only on  $(\alpha, k, n)$ .*

**Quasisymmetric maps.** The proof will use the following standard properties of  $k$ -quasisymmetric maps  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

1. For any ball  $B \subset \mathbb{R}^n$  there exists a ball  $B' \subset \phi(B)$  such that

$$|B'| \asymp |\phi(B)|.$$

2. For any sets  $A \subset B \subset \mathbb{R}^n$  with  $|A| \geq |B|/2$ , we have  $|\phi(A)| \asymp |\phi(B)|$ .
3. More generally, if  $A \subset B$  then

$$\frac{|\phi(A)|}{|\phi(B)|} > \xi_{k,n} \left( \frac{|A|}{|B|} \right) > 0. \quad (2.1)$$

4. The same bounds hold for  $\phi^{-1}$ .

Here the implicit constants and the function  $\xi_{k,n}$  depend only on  $(k, n)$ .

Property (1) is immediate from the definition of quasisymmetry (equation (1.2)), but properties (2), (3) and (4) (for  $n > 1$ ) are not. They can be established by first showing  $\phi$  and  $\phi^{-1}$  are  $k$ -quasiconformal, and then applying distortion estimates for ring domains; see e.g. [Geh2], [Vai1, §18], [AVV, Ch. 14]. Note that property (3) follows by iterating property (2).

**Strategies.** A *positional strategy* for Schmidt's game, or its variant defined by (1.3), is a function  $f$  from the set of balls in  $\mathbb{R}^n$  to itself satisfying  $f(B) \subset B$ . To use this strategy, player  $A$  makes the sequence of moves  $A_i = f(B_i)$ . It can be shown that whenever player  $A$  can win, he can do so using a positional strategy [Sch, Thm. 7].

We say  $f$  is an  $\alpha$ -strategy if  $|f(B)| \geq \alpha|B|$  for all  $B$ .

**Transport.** Let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $k$ -quasisymmetric map. For each topological disk  $D \subset \mathbb{R}^n$ , choose a round disk  $D^\circ \subset D$  of maximal diameter. Note that property (1) above implies

$$|\phi(B)^\circ| \asymp |\phi(B)|$$

for any ball  $B$ , and similarly for  $\phi^{-1}$ .

Using  $\phi$ , we can transport a given positional strategy  $f(B)$  to a new strategy  $f'(B')$  by defining

$$f'(B') = \phi(f(\phi^{-1}(B')^\circ))^\circ. \quad (2.2)$$

**Lemma 2.2** *If  $f$  is an  $\alpha$ -strategy then  $f'$  is an  $\alpha'$ -strategy, where  $\alpha'$  depends only on  $(k, n, \alpha)$ .*

**Proof.** Let  $B = \phi^{-1}(B')^\circ$ , let  $A = f(B) \subset B$ , and let  $A' = \phi(A)^\circ = f'(B')$ . Then by the above properties of  $\phi$  we have

$$\frac{|A'|}{|B'|} \asymp \frac{|\phi(A)|}{|\phi(B)|} \geq \xi_{k,n} \left( \frac{|A|}{|B|} \right) = \xi_{k,n}(\alpha) > 0.$$

Thus we can take  $\alpha' \asymp \xi_{k,n}(\alpha)$ . ■

**Proof of Theorem 2.1** Let  $E' = \phi(E)$ , and consider  $\beta' \in (0, 1)$ .

Let  $f(B)$  denote an  $(\alpha, \beta)$ -strong winning positional strategy for  $E$ , where  $\beta$  is yet to be determined. Let  $f'(B')$  be the transported strategy defined by (2.2). By the preceding Lemma,  $f'$  is an  $\alpha' = \alpha'(k, n, \alpha)$  strategy.

Let  $B'_1, B'_2, \dots$  denote a sequence of moves by player  $B$ , and let  $A'_i = f'(B'_i)$ . We then have

$$|A'_i| \geq \alpha'|B'_i| \quad \text{and} \quad |B'_{i+1}| \geq \beta'|A'_i|.$$

Let  $B_i = \phi^{-1}(B'_i)^\circ$  and let  $A_i = f(B_i)$ , so  $A'_i = \phi(A_i)^\circ$ . Then  $|A_i| \geq \alpha|B_i|$  for all  $i$ . Moreover, there exists a  $\beta > 0$  such that

$$|B_{i+1}| \geq \beta|A_i| \tag{2.3}$$

for all  $i$ . Indeed, equation (2.1) implies

$$\frac{|B_{i+1}|}{|A_i|} \asymp \frac{|\phi^{-1}(B'_{i+1})|}{|\phi^{-1}(A'_i)|} \geq \xi_{k,n} \left( \frac{|B'_{i+1}|}{|A'_i|} \right) = \xi_{k,n}(\beta') > 0,$$

so (2.3) holds with  $\beta \asymp \beta'$ .

Since  $f$  is  $(\alpha, \beta)$ -strong winning, the intersection  $\bigcap A_i = \bigcap B_i$  meets  $E$ , and hence  $\phi(\bigcap B_i) \subset \bigcap B'_i$  meets  $\phi(E) = E'$ . Thus  $f'$  is winning strategy for player  $A$ . Since  $\beta' \in (0, 1)$  was arbitrary, this shows  $E'$  is an  $\alpha'$ -strong winning set.  $\blacksquare$

**Notes and references.** We remark that Schmidt's game and its variants can be played in general metric spaces (cf. [Sch]), and Theorem 1.2 holds in this setting as well, provided quasisymmetric maps are defined in such a way that properties (1-4) above hold (they are no longer formal consequences of equation (1.2)). For more on quasiconformal maps in general metric spaces, see e.g. [TV], [HK].

Note that for  $n \geq 2$ , quasiconformal homeomorphism of  $\mathbb{R}^n$  automatically preserve sets of measure zero and sets of dimension  $n$  [Vai1, §33], [Geh1, Thm.3].

### 3 Diophantine sets

In this section we prove the Diophantine set  $D(\Gamma)$  is strong winning (Theorem 1.3).

**Hyperbolic space.** Let  $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$  denote the upper halfspace with coordinates  $(x, y)$ , endowed with the hyperbolic metric

$$\rho^2 = \frac{|dx|^2 + |dy|^2}{y^2}$$

of constant curvature  $-1$ . We will identify  $\partial\mathbb{H}^{n+1}$  with the sphere  $\mathbb{R}^n \cup \{\infty\}$ . For any ball  $B = B(x, r) \subset \mathbb{R}^n$ , we let

$$\text{top}(B) = (x, r) \in \mathbb{H}^{n+1}.$$

The top of  $B$  is the highest point on the hyperbolic plane spanned by  $\partial B$ . Each  $x \in \mathbb{R}^n$  determines a vertical geodesic

$$\gamma_x(t) = (x, e^{-t}) : \mathbb{R} \rightarrow \mathbb{H}^{n+1},$$

parameterized by arclength and satisfying  $\gamma_x(t) \rightarrow x$  as  $t \rightarrow \infty$ .

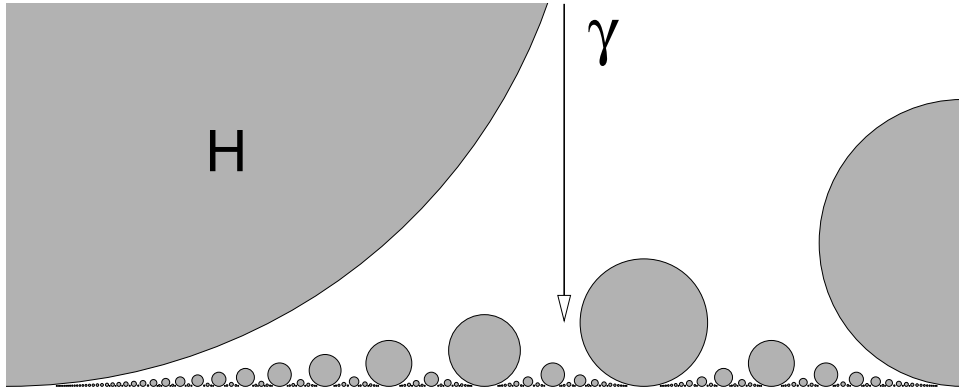


Figure 1. A bounded geodesic avoids an infinite array of horoballs.

**Bounded geodesics.** Now let  $\Gamma \subset \text{Isom}(\mathbb{H}^{n+1})$  be a lattice with a cusp at infinity, and let

$$\pi : \mathbb{H}^{n+1} \rightarrow M = \mathbb{H}^{n+1}/\Gamma$$

denote the projection to its finite-volume quotient space. The associated Diophantine set is defined by

$$D(\Gamma) = \{x \in \mathbb{R}^n : \pi(\gamma_x(t)) \text{ remains bounded as } t \rightarrow +\infty\};$$

equivalently,  $\overline{\pi(\gamma_x(0, \infty))} \subset M$  is compact.

**Horoballs and cusps.** A *horoball*  $H \subset \mathbb{H}^{n+1}$  is a hyperbolic ball of ‘infinite radius’, centered at a point  $p \in \partial\mathbb{H}^{n+1}$ . A horoball centered at  $p \in \mathbb{R}^n$  is simply a Euclidean ball of the form

$$H = H(p, r) = B((p, r), r) \subset \mathbb{R}^{n+1}.$$

A horoball centered at  $p = \infty$  is a halfspace of the form  $H(\infty, r) = \{(x, y) : y > 1/r\}$ . We will use the shorthand  $\epsilon H$  for  $H(p, \epsilon r)$  when

$H = H(p, r)$ . Note that  $\partial H$  and  $\partial(\epsilon H)$  are parallel in the hyperbolic metric, with separation  $|\log \epsilon|$ .

The basic structure theorems for finite-volume hyperbolic manifolds imply there is a  $\Gamma$ -invariant collection of disjoint horoballs  $\mathcal{H}'$  such that

$$M_\epsilon = \overline{\mathbb{H}^{n+1} - \bigcup \{\epsilon H : H \in \mathcal{H}'\}} / \Gamma$$

is compact for every  $\epsilon > 0$ , and  $M = \bigcup_{\epsilon > 0} M_\epsilon$ . There is a unique infinite horoball  $H(\infty, r) \in \mathcal{H}'$ ; we let  $\mathcal{H} \subset \mathcal{H}'$  denote the remaining set of finite horoballs. The Diophantine set  $D(\Gamma)$  can then be described as follows:

*A point  $x \in \mathbb{R}^n$  belongs to  $D(\Gamma) \iff$  There is an  $\epsilon > 0$  such that  $\gamma_x(t)$  avoids  $\epsilon H$  for all  $H \in \mathcal{H}$  and  $t > 0$ .*

See Figure 1.

**The game.** Let us now analyze a typical game

$$B_1 \supset A_1 \supset B_2 \supset A_2 \dots$$

played using the modified rule

$$|A_i| \geq \alpha |B_i| \quad \text{and} \quad |B_{i+1}| \geq \beta |A_i|.$$

Suppose for simplicity that  $\bigcap A_i$  consists of a single point  $x$ . Let  $B_i = B(x_i, r_i)$  and let  $t_i = -\log r_i$ . Since  $|x - x_i| \leq r_i$ , the points  $\text{top}(B_i) = (x, r_i)$  and points  $\gamma_x(t_i) = (x, r_i)$  satisfy

$$d(\gamma_x(t_i), \text{top}(B_i)) \leq 1 \tag{3.1}$$

in the hyperbolic metric. We also have

$$|t_i - t_{i+1}| = \log(|B_i|/|B_{i+1}|) \leq |\log \alpha \beta|. \tag{3.2}$$

Thus the sequence of points  $\text{top}(B_i)$  is a good approximation to the geodesic  $\gamma_x$ . In particular, these points record the horocyclic excursions of  $\gamma_x$ .

**Lemma 3.1** *Suppose  $H = H(p, r) \in \mathcal{H}$  and  $\gamma_x(t) \in \epsilon H$  for some  $t > t_1$ , where  $\epsilon = \alpha\beta/e$ . Then there is an  $i \geq 1$  such that  $\text{top}(B_i) \in H$ . Moreover,  $i$  can be chosen so that  $\epsilon r \leq |B_i|$  if  $i \neq 1$ .*

**Proof.** Let  $H' = (1/e)H$ , and let  $\gamma_x^{-1}(H') = (a, b) \subset \mathbb{R}$ . Then for any  $i \geq 1$ ,

$$t_i \in (a, b) \implies \text{top}(B_i) \in H,$$

since  $d(\partial H, \partial H') = 1$  and  $d(\gamma_x(t_i), \text{top}(B_i)) \leq 1$ . We also have

$$|a - b| \geq d(\partial H', \partial(\epsilon H)) = |\log \alpha \beta|,$$

since the path  $\gamma_x(a, b)$  joins  $\partial H'$  to  $\partial \epsilon H$ .

Since  $t_1 < t \in [a, b]$ , we have  $b > t_1$ . If  $t_1 \geq a$  then  $\text{top}(B_1) \in H$ , and the proof is complete.

Otherwise  $t_1 < a$ . Since  $t_i \rightarrow \infty$  with  $|t_i - t_{i+1}| \leq b - a$  (by equation (3.2), there exists an  $i$  such that  $t_i \in [a, b]$ . This implies  $\text{top}(B_i) \in H$ . Moreover, we have  $t_i - a \leq |\log \alpha \beta|$ . Since a vertical geodesic meeting  $H'$  must cross the horizontal diameter of  $H'$ , we also have  $r/e \leq e^{-a}$  and hence

$$\epsilon r = \alpha \beta r / e \leq \alpha \beta e^{-a} \leq e^{-t_i} = r_i = |B_i|$$

as desired. ■

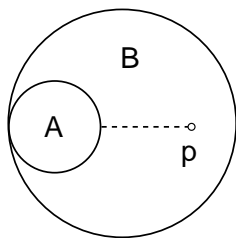


Figure 2. The strategy  $A = f(B)$  when  $\text{top}(B)$  meets  $H(p, r)$ .

**The strategy.** Now fix  $\alpha \in (0, 1/2)$ . We define an  $\alpha$ -positional strategy  $A = f(B) \subset B$  as follows:

1. If  $\text{top}(B) \in H = H(p, r) \in \mathcal{H}$ , then we choose  $A \subset B$  with  $|A| = \alpha|B|$  such that  $A$  is as far away from  $p$  as possible (see Figure 2).
2. Otherwise, we choose  $A \subset B$  with  $|A| = \alpha|B|$  so that  $A$  has the same center as  $B$ .

**Lemma 3.2** *If  $x \in A = f(B)$  and  $\text{top}(B)$  meets  $H = H(p, r) \in \mathcal{H}$ , then  $\gamma_x$  avoids  $H(p, s)$  where  $s = (1/2 - \alpha)|B|$ .*

**Proof.** Since  $A$  is tangent to  $\partial B = B(x, r)$ , it is disjoint from  $B(x, s)$  where  $s = r - |A| = (1/2 - \alpha)|B|$ . By construction,  $A$  is also disjoint from  $B(p, s)$ , which implies any vertical ray landing in  $A$  avoids  $H(p, s)$ . ■

Theorem 1.3 now follows from:

**Theorem 3.3** *The  $\alpha$ -strategy  $f$  is strong winning for every  $\beta \in (0, 1)$ .*

**Proof.** Consider any game played with  $A_i = f(B_i)$  and  $|B_{i+1}| \geq \beta|A_i|$ . Since  $|A_{i+1}| \leq \alpha|A_i|$ , there is a unique point  $x$  in  $\bigcap A_i$ . To complete the proof, we will show there is a  $\delta > 0$  such that  $\gamma_x(t)$  avoids  $\bigcup_{\mathcal{H}} \delta H$  for all  $t > t_1$ . (This is equivalent to showing  $x \in D(\Gamma)$ .)

Let  $\epsilon = \alpha\beta/e$ , and let

$$\mathcal{H}_x = \{H \in \mathcal{H} : \gamma_x \text{ meets } \epsilon H\}.$$

By Lemma 3.1, for each  $H \in \mathcal{H}_x$  there is an  $i \geq 1$  such that  $\text{top}(B_i) \in H = H(p, r)$  and  $|B_i|/r \geq \epsilon$ . Let us use the least such  $i$  to label  $H = H_i = H_i(p_i, r_i)$ , and let  $I$  denote the set of indices so obtained; then  $\mathcal{H}_x = \{H_i : i \in I\}$ .

Lemma 3.2 then implies that  $\gamma_x$  avoids  $\delta_i H_i$ , where

$$\delta_i = (1/2 - \alpha)|B_i|/r_i \geq (1/2 - \alpha)\epsilon \quad (3.3)$$

Thus  $\gamma_x$  avoids  $\bigcup_{\mathcal{H}} \delta H$  for

$$\delta = \min(\delta_1, (1/2 - \alpha)\epsilon),$$

and the proof is complete. ■

**Remarks.** Note that the strategy used by player  $A$  does not depend on the value of  $\beta$ , and that all of  $A$ 's moves satisfy  $|A_i| = \alpha|B_i|$ .

Note also that the geodesic  $\pi(\gamma_x) \subset M$  produced by player  $A$  lies mostly in a set of diameter  $O(|\log \beta|)$ . More precisely, if we fix  $\alpha = 1/4$ , then the bound (3.3) implies that  $\gamma_x$  meets only finitely many horoballs in  $\bigcup (\beta/200)H$ . Thus  $\pi(\gamma_x) \subset M$  lies in a set of diameter  $2|\log \beta| + O(1)$  — apart from some initial excursions into the cusps which can be forced by  $B$ 's first move.

**The steering game.** Here is related game which can be played in hyperbolic space. Two players take turns choosing smooth segments in hyperbolic space satisfying

$$|A_i| \leq a \quad \text{and} \quad |B_i| \leq b$$

for fixed  $a, b > 0$ , such that the segments fit together to form a smooth ray

$$\gamma = B_1 \cup A_1 \cup B_2 \cup A_2 \dots \subset \mathbb{H}^{n+1}$$

with geodesic curvature  $< 1/2$ . Then  $\gamma$  is a quasigeodesic, so it converges to a unique point  $x \in \partial\mathbb{H}^{n+1}$  (provided its length is infinite). We say  $E \subset \mathbb{R}^n$  is winning for the *steering game* if there is an  $\alpha > 0$  such that for all  $\beta > 0$ , player  $A$  can insure that  $x \in E$ .

It is straightforward to show that  $E$  is winning for the steering game iff  $E$  is strong winning. Thus Theorem 1.3 can be formulated informally as follows: if  $A$  is driving a car at (high) uniform speed in a finite-volume hyperbolic manifold  $M$ , but he can only control the wheel for one minute every hour, then he can still keep the car within a bounded distance from home (provided the car makes no sharp turns.)

## 4 Counterexamples

Given  $x \in \mathbb{R}$ , we will write the fractional part of  $x$  in base 2 as  $\{x\} = 0.x_1x_2x_3\dots$ , and set  $Z(x) = \{s : x_s = 0\}$ . Let

$$W = \{x \in \mathbb{R} : Z(x) \text{ contains an almost arithmetic progression}\}.$$

Here an *almost arithmetic progression* is a set of the form

$$P = \lfloor \sigma + \mathbb{N}\theta \rfloor \subset \mathbb{Z},$$

where  $\sigma, \theta \in \mathbb{R}$  and  $\theta > 0$ .

In this section we will show:

**Theorem 4.1** *The set  $W$  is winning but not strong winning.*

**Theorem 4.2** *For every  $k > 1$  there is a  $k$ -quasisymmetric map  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\phi(W)$  is not winning.*

**Winning.** We begin the proof of Theorem 4.1 by showing:

**Proposition 4.3** *The set  $W$  is  $\alpha$ -winning for all  $\alpha < 1/8$ .*

**Proof.** Note that  $|B_{i+1}| = (\alpha\beta)^i |B_1|$  for all  $i > 0$ . Choose  $\sigma, \theta$  so that  $2^{-\theta} = \alpha\beta$  and  $2^{-\sigma-\theta} = (1/4)|B_1|$ ; then we have

$$2^{-\sigma-i\theta} = (1/4)|B_i|$$

for all  $i$ . Thus the integers  $s_i = \lfloor \sigma + i\theta \rfloor$  form an almost arithmetic progression that satisfies

$$(1/2)|B_i| \geq 2^{-s_i} \geq (1/4)|B_i| > 2|A_i|$$

for all  $i$ . These bounds imply player  $A$  can choose  $A_i$  such that  $x_{s_i} = 0$  for all  $x \in A_i$ . Then the unique point  $x \in \bigcap A_i$  will belong to  $W$ , since  $\{s_1, s_2, \dots\} \subset Z(x)$ . ■

To show  $W$  is not strong winning, we first note:

**Lemma 4.4** *Suppose  $0 < \beta < \alpha^{100}$ . Then in the  $(\alpha, \beta)$ -Schmidt game, player  $B$  can force the unique point  $x \in \bigcap B_i$  to satisfy*

$$Z(x) \subset Z_\theta = B(\lfloor \mathbb{N}\theta \rfloor, \theta/10),$$

where  $\theta = -\log_2(\alpha\beta)$ .

**Proof.** Player  $B$  will use the following strategy: he begins the game by choosing  $B_1 = [0, 1]$ , and thereafter chooses  $B_i$  to force  $x_j = 1$  as often as possible. We may assume  $\alpha < 1/2$ , since otherwise player  $B$  has complete control over  $x$ . Then, since  $|B_{i+1}| = (\alpha\beta)^i = 2^{-i\theta}$ , while  $|A_{i+1}| = \alpha|B_{i+1}| > 2^{-i\theta-\theta/20}$ , player  $A$  can only control the digit  $x_j$  if  $j \in Z_\theta$ , and player  $B$ 's strategy forces  $x_j = 1$  for all remaining  $j$ . ■

**Lemma 4.5** *Suppose  $P = \lfloor \sigma + \mathbb{N}\theta \rfloor$  is close to  $P' = \lfloor \sigma' + \mathbb{N}\theta' \rfloor$ , in the sense that*

$$P \subset B(P', \theta'/6 - 2).$$

*Then  $\theta \in \mathbb{N}\theta'$ .*

**Proof.** The hypotheses imply that

$$\mathbb{N}(\theta/\theta') \subset B(\mathbb{N} + \delta, 1/6),$$

where  $\delta = (\sigma' - \sigma)/\theta'$ . Hence  $G = \mathbb{N}(\theta/\theta') \bmod 1$  is contained in an interval of length less than  $< 1/3$  on  $S^1 = \mathbb{R}/\mathbb{Z}$ . But there is no nontrivial semigroup contained in such a short interval, so  $G = \{0\}$  and  $\theta/\theta' \in \mathbb{Z}$ . ■

**Proof of Theorem 4.1.** We will show that for each  $\alpha \in (0, 1)$ , the set  $W$  is not strong winning for  $\beta = \alpha^{200}$ . We may assume  $\alpha < 1/2$ .

First note that, upon replacing  $\beta$  with  $\beta/\alpha$ , we can assume  $|A_i| = \alpha|B_i|$  for every  $i$ . If player  $A$  does not choose  $|A_i|$  this small, player  $B$  can choose an arbitrary  $A'_i \subset A_i$  with  $|A'_i| = \alpha|A_i|$  and then respond to this move instead.

Now for any  $\beta' \in [\alpha^{100}, \alpha^{101}]$ , the moves for the  $(\alpha, \beta')$ -Schmidt game are also legal for the modified  $(\alpha, \beta)$  game. Thus player  $B$  can follow the strategy provided by Lemma 4.4 to insure

$$Z(x) \subset B(\lfloor \mathbb{N}\theta' \rfloor, \theta'/10),$$

where  $\theta' = -\log_2(\alpha/\beta') > 100$ . Thus if  $Z(x)$  contains an almost arithmetic progression  $\lfloor \sigma + \mathbb{N}\theta \rfloor$ , Lemma 4.5 guarantees that

$$\theta \in \mathbb{N}\theta' \tag{4.1}$$

(since  $\theta'/10 < \theta'/6 - 2$ ).

Player  $B$  has a similar strategy that guarantees

$$\theta \in \mathbb{N}\theta'', \tag{4.2}$$

for any  $\theta''$  close enough to  $\theta'$ . By alternating between longer and longer runs of each of these two strategies, player  $B$  can insure that both (4.1) and (4.2) hold. To win the game, player  $B$  simply chooses  $\beta'$  and  $\beta''$  such that  $\theta''/\theta'$  is irrational. Then  $(\mathbb{N}\theta') \cap (\mathbb{N}\theta'') = \{0\}$ ; therefore  $Z(x)$  contains no almost arithmetic progression, and hence  $x \notin W$ . ■

**Quasisymmetric maps.** We now turn to the proof of Theorem 4.2. The idea of the proof is that, by applying a quasisymmetric map to  $W$ , we can essentially allow player  $B$  to vary the value of  $\beta$ . Since  $W$  is not strong winning,  $\phi$  can be chosen so  $\phi(W)$  is not winning.

To get some breathing room, we will write  $W = \bigcup_{T=1}^{\infty} W_T$ , where

$$W_T = \{x \in \mathbb{R} : Z(x) \supset [\sigma + \mathbb{N}\theta] \text{ for some } \sigma, \theta \in [1, T]\}.$$

It is easy to see that  $W_T \cap [0, 1]$  is a closed, porous set, and thus:

**Proposition 4.6** *The winning set  $W$  is a  $\sigma$ -porous set.*

The following tool will be useful in the construction of  $\phi$ .

**Lemma 4.7** *Let  $E \subset [0, 1]$  be a porous set. Then there is a family of  $k(u)$ -quasisymmetric maps  $\phi_u : [0, 1] \rightarrow [0, 1]$  defined on a neighborhood of  $u = 1$ , such that*

$$(1/2)|B|^u < |\phi_u(B)| < 2|B|^u$$

for all balls  $B \subset [0, 1]$  meeting  $E$ , and  $k(u) \rightarrow 1$  as  $u \rightarrow 1$ .

**Idea of the proof.** One can construct  $\phi_u = \cdots \phi_3 \circ \phi_2 \circ \phi_1$  by composing an infinite sequence of smooth quasisymmetric maps where  $\phi'_i(x) = 2^{1-u}$  at scale  $2^{-i}$  near  $E$ , and  $\phi_i$  is nearly the identity at larger scales. By taking advantage of the complementary gaps (provided by porosity), one can insure that the composition is quasisymmetric with  $k(u)$  close to 1. ■

The existence of such a  $\phi$  is not surprising, since a  $k$ -quasisymmetric map need only be  $(1/k)$ -Hölder continuous. Such maps arise naturally as conjugacies in 1-dimensional dynamics.

**Shifting strategies.** Now fix  $\alpha' \in (0, 1/2)$  and set  $\beta' = (\alpha')^{200}$ .

Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a  $k$ -quasisymmetric map. Consider a sequence of moves  $(A'_i, B'_i)$  for the  $(\alpha', \beta')$ -Schmidt game, with  $x'$  the unique point in  $\bigcap A'_i$ . The preimages of  $A'_i, B'_i$  and  $x'$  under  $\phi$  will be denoted by  $A_i, B_i$  and  $x$ .

We will present a sequence of results leading up to a construction of a mapping  $\phi$  such that  $W' = \phi(W)$  is not  $(\alpha', \beta')$ -winning.

The strategy for player  $B$  will always be the same; we call it the *basic strategy*. First, player  $B$  chooses  $B'_1$  so that  $B_1 = [0, 1]$ . Then, for each  $i > 1$ , player  $B$  chooses  $B'_i$  so that  $x_j = 1$  for all  $x \in B_i$  and all  $j$  in the range he can control. This range is essentially

$$J_i = \{j : -\log_2 |A_{i-1}| < j < -\log_2 |B_i|\}.$$

A key point is that  $J_i$  depends on  $\phi$ ; we have  $|B'_i| = (\alpha\beta)^{i-1}$ , but  $|B_i|/|B'_i|$  may be large or small. On the other hand, if  $k$  is close to

1, then  $|B_i|$  is close to  $\beta'|A_{i-1}|$ , so the number of  $j$  that player  $B$  can control with a single move (the length of  $J_i$ ) is still close to  $-\log_2 \beta'$ .

Let  $\theta' = -\log_2(\alpha'\beta')$ .

**Lemma 4.8** *Fix  $T$  and  $k > 1$ . Then for  $\theta$  close enough to  $\theta'$ , there is a  $k$ -quasisymmetric map  $\phi$  such that if  $B$  plays the basic strategy, and  $x \in W_T \cap \bigcap B_i$ , then*

$$Z(x) \subset B(\lfloor N\theta \rfloor, \theta/10). \quad (4.3)$$

**Proof.** Observe that the strategy for  $B$  employed in the proof Lemma 4.4 is simply the basic strategy in the case  $\phi(x) = x$ . Thus Lemma 4.4 gives the result above in the special case  $\theta = \theta'$ .

For the general case, let  $u = \theta'/\theta$ , and apply Lemma 4.7 to construct a map  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  with the property that  $|A_i|^u \asymp |A'_i|$  and  $|B_i|^u \asymp |B'_i|$  whenever  $x \in W_T$ . This means  $(A_i, B_i)$  are essentially moves in the  $(\alpha, \beta)$ -Schmidt game, where  $(\alpha^u, \beta^u) = (\alpha', \beta')$ . The basic strategy for player  $B$  then produces a sequence of moves  $B_i$  of the same type used in the proof of Lemma 4.4; and this insures the inclusion (4.3) holds whenever  $x \in W_T$ . ■

**Lemma 4.9** *For all  $T$  and  $k > 1$ , there exists a  $k$ -quasisymmetric  $\phi$  such that  $\phi(W_T)$  is not  $(\alpha', \beta')$ -winning.*

**Proof.** To construct  $\phi$ , choose  $\theta''$  close to  $\theta'$  such that their ratio is irrational. Then there is an  $r > 0$  such that

$$B(\lfloor N\theta' \rfloor, r) \cap B(\lfloor N\theta'' \rfloor, r) \cap [0, T] = \emptyset. \quad (4.4)$$

Let  $u = \theta''/\theta'$ . Choose a large integer  $N$  and, using Lemma 4.7, construct a  $k$ -quasisymmetric mapping  $\phi$  such that

1.  $\phi(x)_j = x_j$  for all  $j \leq N$ ; and
2.  $(2^N |\phi(B)|)^u \asymp (2^N |B|)$  for any ball  $B$  meeting  $W_T$  with  $2^{-2N} < |B| < 2^{-N}$ .

Since  $u$  is close to 1, we can also arrange that  $k$  is close to 1, independent of  $N$ .

Let  $Z_N(x) = Z(x) \cap [0, N]$ . Because of (1), the basic strategy for player  $B$  forces

$$Z_N(x) \subset B(\lfloor N\theta' \rfloor, \theta'/10).$$

On the other hand, because of (2), it also forces

$$Z_{2N}(x) \setminus Z_N(x) \subset B(\lfloor N\theta'' \rfloor, \theta''/10).$$

Now suppose  $x \in W_T$ . Then  $Z(x) \supset [\sigma + \mathbb{N}\theta]$  for some  $\sigma, \theta \leq T$ . For  $N$  sufficiently large, the two conditions above insure that  $\theta$  is within distance  $r$  of both  $[\mathbb{N}\theta']$  and  $[\mathbb{N}\theta'']$ . But there is no such  $\theta$ , by equation (4.4).

Thus player  $B$  wins if he uses the basic strategy; indeed, his win is guaranteed by the time he chooses  $B_{2N}$ .  $\blacksquare$

**Final jeopardy: Proof of Theorem 4.2.** Given  $k > 1$ , we will construct a  $k$ -quasisymmetric map such that  $W' = \phi(W)$  is not winning. The map will be normalized so that  $\phi[0, 1] = [0, 1]$ .

Player  $B$  will use the basic strategy. This strategy can be carried out so there are only finitely many choices for each  $B_i$ , and each choice satisfies

$$W \cap \partial B_i = \emptyset.$$

This last condition is easy to achieve since  $W$  is nowhere dense.

Along with  $\phi$  we will construct a sequence of disjoint intervals  $N_{n,T} \subset \mathbb{N}$ , indexed by integers  $n, T > 0$ , such that whenever  $B$  plays the basic strategy for the  $(\alpha', \beta')$ -game, we have:

$$(\alpha', \beta') = (2^{-n}, 2^{-200n}) \implies W_T \cap \bigcap_{i \in N_{n,T}} B_i = \emptyset. \quad (4.5)$$

This suffices to complete the proof. Indeed, if  $\phi(W)$  is winning, then it is  $(2^{-n}, 2^{-200n})$ -winning for some  $n > 0$  (cf. [Sch, Lemma 11]); but (4.5) implies that, if  $B$  follows the basic strategy, then the unique point  $x \in \bigcap B_i$  will not belong to  $\bigcup W_T = W$ .

To construct  $\phi$ , let  $(n_j, T_j)$ ,  $j = 1, 2, 3, \dots$  enumerate all the possible pairs  $(n, T)$  as above. Let  $(\alpha_j, \beta_j) = (2^{-n_j}, 2^{-200n_j})$ , and choose  $k_j > 1$  such that  $\prod k_j = k$ .

By Lemma 4.9, there is a  $k_1$ -quasisymmetric map such that  $\phi_1(W_{T_1})$  is not  $(\alpha_1, \beta_1)$ -winning. In fact (as we have seen in the proof),  $\phi_1$  can be chosen so that player  $B$  wins after a finite set of moves, ranging in a finite interval  $N_{n_1, T_1} \subset \mathbb{N}$ .

Then  $\phi = \phi_1$  yields (4.5) for  $(n, T) = (n_1, T_1)$ . Let  $F_1 \subset [0, 1]$  be the finite set of possible endpoints for  $B_i$  with  $i \in N_{n_1, T_1}$ .

Since  $F_1$  is finite, we may also arrange that  $\phi_1$  is differentiable; it can be smoothed at a small scale without changing the values of  $\phi(x)$ ,  $x \in F_1$ , and hence without changing (4.5) for  $(n_1, T_1)$ .

By differentiability, the compact set  $\phi_1(W_{T_2}) \cap [0, 1]$  can be covered by a collection of disjoint intervals  $\mathcal{I}_2$  which each meet  $\phi_1(W_{T_2})$  in a nearly linearly rescaled piece of  $W_{T_2}$ . Since  $\phi_1(F_1)$  is disjoint from  $\phi_1(W) \supset \phi_1(W_{T_2})$ , we can also assume that  $d(I, \phi_1(F_1)) \gg |I|$  for each  $I \in \mathcal{I}_2$ .

By applying Lemma 4.9 in each of these intervals and assembling the results, we obtain a  $k_2$ -quasisymmetric mapping  $\phi_2$  such that  $W'_2 =$

$\phi_2(\phi_1(W_{T_2}))$  is not  $(\alpha_2, \beta_2)$ -winning. Moreover, we can find an interval  $N_{n_2, T_2}$  disjoint from  $N_{n_1, T_2}$  such that player  $B$ 's moves in this range already exclude  $x$  from  $W_{T_2}$ . Finally, we can arrange that  $\phi_2(x) = x$  for all  $x \in \phi_1(F_1)$ .

Then  $\phi = \phi_2 \circ \phi_1$  yields (4.5) for both  $(n, T) = (n_1, T_1)$  and  $(n, T) = (n_2, T_2)$ . Continuing in the same fashion, we obtain in the limit a mapping

$$\phi = \lim_{j \rightarrow \infty} \phi_j \circ \phi_{j-1} \circ \cdots \circ \phi_2 \circ \phi_1$$

such that (4.5) holds for all  $n, T \geq 1$ . The limit is  $k$ -quasisymmetric because  $\prod k_j = k$ .  $\blacksquare$

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