

# Amenable coverings of complex manifolds and holomorphic probability measures

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## 1 Introduction.

Let  $\pi : Y \rightarrow X$  be a covering map between complex manifolds  $X$  and  $Y$ . For many holomorphic objects (such as functions or forms) one can define a pushforward operator  $\pi_*$  carrying objects on  $Y$  to objects on  $X$  by summing over the fiber. The pushforward is *efficient* if there is little cancellation in the sum. To achieve efficiency, there must be coherence in phase on different sheets of the covering.

In this paper we will demonstrate a direct connection between *efficiency* of the push-forward  $\pi_*$  and *amenability* of the covering  $\pi$ . The latter is a purely combinatorial property of the pair of groups  $\pi_1(Y) \subset \pi_1(X)$ ; the covering is amenable if there exists a  $\pi_1(X)$ -invariant finitely additive probability measure on the coset space  $\pi_1(X)/\pi_1(Y)$ . Amenability is discussed in more detail in §3.

The connection will be established by studying complex-valued measures on the fibers of the covering, varying holomorphically with respect to the base.

**Motivating Example.** To any Riemann surface  $X$  we can associate the Banach space  $Q(X)$  of holomorphic quadratic differentials  $\phi(z)dz^2$  such that

$$\|\phi\| = \int_X |\phi| < \infty.$$

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Then associated to a covering  $Y \rightarrow X$  there is an operator

$$\Theta_{Y/X} : Q(Y) \rightarrow Q(X)$$

given by  $\phi \mapsto \pi_*(\phi)$ . It is easy to see that  $\|\Theta_{Y/X}\| \leq 1$ .

Let  $B_X$  and  $B_Y$  denote the open unit balls in  $Q(X)$  and  $Q(Y)$  respectively.

**Theorem 1.1 ([Mc1])** *Let  $Y \rightarrow X$  be a covering of a hyperbolic Riemann surface  $X$ . Either:*

1. *The covering is amenable, and  $\Theta(B_Y) = B_X$ , or*
2. *The covering is nonamenable, and  $\overline{\Theta(B_Y)}$  is contained in the interior of  $B_X$ .*

**Corollary 1.2 (Kra's Theta conjecture)** *For the universal cover  $\pi : \Delta \rightarrow X$  of a hyperbolic surface of finite area,*

$$\|\Theta_{\Delta/X}\| < 1.$$

In other words, there is always a definite inefficiency in the representation of  $\phi$  as a pushforward  $\pi_*(\psi)$ .

**Remark.** Let  $\Gamma$  be the Fuchsian group of deck transformations for the universal covering  $\Delta \rightarrow X$ . Then given  $\psi \in Q(\Delta)$ , its pushforward can be expressed by the classical Poincaré series [Poin]

$$\Theta(\psi) = \sum_{\gamma \in \Gamma} \gamma^*(\psi).$$

This sum is clearly  $\Gamma$ -invariant, so it defines an element of  $Q(X)$ .

**Proof of the corollary.** For a surface of finite hyperbolic area, it is easy to see that the universal cover is nonamenable (cf. [Mc1]), and it is well-known that  $Q(X)$  is finite dimensional. Thus  $\overline{\Theta(B_\Delta)}$  is a compact subset of  $B_X$ , so  $\|\Theta\| < 1$ . ■

These results have implications for Teichmüller theory and for the construction of hyperbolic structures on 3-manifolds; see [Mc4], [Mc3] for an expository account and [Mc1], [Mc2] for details.

Recently Barrett and Diller gave an elegant and surprising new proof of Theorem 1.1 in the case of the universal covering (and thus a new proof of the Theta conjecture), as part of their work on affine bundles and holomorphic

averaging [BD]. Here we present an understanding of their argument which applies to all coverings, not just the universal covering, and which leads to a short proof of the full Theorem 1.1.

**Outline of the paper.** §2 states the main result of this paper (Theorem 2.1, Efficiency implies amenability) and applies it to give a new proof of Theorem 1.1. Theorem 2.1 is proved in §3. We conclude in §4 with a geometric sketch of the Barrett-Diller proof of the Theta conjecture, to help indicate the connection between their ideas and ours.

**Remark.** Theorem 1.1 is not quite sufficient for the applications of [Mc2]. For example, we need a uniform bound on  $|\Theta_{Y_n/X}|$  for a certainly countable collection of covering spaces  $Y_n$ . The proof in [Mc1] has the advantage of yielding an effective bound on  $|\Theta|$ .

## 2 Statement of results

To set up the main theorem we require:

- $\pi : Y \rightarrow X$  : a covering map between (connected) complex manifolds;
- $\mathcal{L} \rightarrow X$ : a holomorphic line-bundle over  $X$ ;
- $\mathcal{L}' \rightarrow Y$ : the pull-back of  $\mathcal{L}$  to  $Y$ ;
- $\phi : X \rightarrow \mathcal{L}$ : a holomorphic section of  $\mathcal{L}$ , *not identically zero*; and
- $\psi_n : Y \rightarrow \mathcal{L}'$ : a sequence of holomorphic sections such that  $\pi_*(\psi_n) = \phi$ .

**Pushforward.** Here is a more precise explanation of what we mean by  $\pi_*(\psi) = \phi$ . Let  $U \subset X$  be any open ball. Then  $\pi^{-1}(U)$  is a collection  $V_i$  of disjoint balls in  $Y$ , and  $\pi$  admits a holomorphic inverse  $\rho_i : U \rightarrow V_i$  for each  $i$ . If  $\sum \rho_i^*(\psi)$  converges to  $\phi$  uniformly on compact subsets of  $U$ , for every such  $U$ , then we say  $\pi_*(\psi) = \phi$ .

**Efficiency.** The sequence  $\psi_n$  is *efficient* if  $\pi_*|\psi_n|$  converges to  $|\phi|$  uniformly on compact subsets of  $X$ .

Here  $|\phi|$  is a section of the oriented real line-bundle  $|\mathcal{L}|$  naturally attached to  $\mathcal{L}$ . In down-to-earth terms, one can trivialize  $\mathcal{L}$  over a ball  $U \subset X$ , so the  $\psi$  and  $\phi$  become functions; then efficiency means  $\sum |\rho_i^*(\psi_n)|$  is close to  $|\phi|$  when  $n$  is large.

**Theorem 2.1 (Efficiency implies amenability)** *If  $\pi : Y \rightarrow X$  admits an efficient sequence  $\psi_n : Y \rightarrow \mathcal{L}'$  with  $\pi_*(\psi_n) = \phi$ , then the covering  $\pi$  is amenable.*

We record the following complementary result:

**Theorem 2.2** *If  $\pi : Y \rightarrow X$  is amenable and  $\phi = \pi_*(\psi)$  for some  $\psi : Y \rightarrow \mathcal{L}'$ , then there is an efficient sequence  $\psi_n$  with  $\pi_*(\psi_n) = \phi$  and  $\pi_*|\psi_n| \leq \pi_*|\psi|$ .*

When the covering is regular with deck group  $\mathbb{Z}$  generated by  $g$ , the  $\psi_n$  can be given by

$$\psi_n = \frac{1}{n} \sum_{k=1}^n (g^k)^* \psi.$$

For a general amenable covering, the proof of Theorem 2.2 is a straightforward generalization of [Mc1, Theorem 9.1]; we omit the details.

**Remark.** Theorems 2.1 and 2.2 also hold for finite dimensional affine bundles; for simplicity we stick with the versions above. The versions for line bundles have applications to bounded symmetric domains (see the Appendix to [Mc1]).

**Proof of Theorem 1.1.** Assuming the results above, we present a new proof of the main facts concerning  $||\Theta||$ .

Let  $\pi : Y \rightarrow X$  be a covering of Riemann surfaces. Quadratic differentials on  $X$  and  $Y$  are sections of bundles  $\mathcal{L}$  and  $\mathcal{L}'$  obtained by squaring the canonical bundles. Note that a section of  $|\mathcal{L}|$  is naturally an area form on  $X$  (since  $|\phi| = |\phi(z)||dz|^2$ ) and that for  $\psi \in Q(Y)$ ,

$$||\psi|| = \int_X \pi_*|\psi_n|.$$

Now suppose the covering is amenable. Let  $\phi$  be a member of  $B_X$ , so  $||\phi|| = 1 - \epsilon < 1$ . By a result of Ahlfors and Bers,  $\phi = \pi_*(\psi)$  for some  $\psi$  in  $Q(Y)$  (see [Kra]). By Theorem 2.2, there is an efficient sequence  $\psi_n$  with  $\pi_*(\psi_n) = \phi$  and  $\pi_*|\psi_n| \leq \pi_*|\psi|$ . Since the latter is integrable, there is compact set  $K \subset X$  such that  $\int_{X-K} \pi_*|\psi_n| < \epsilon/2$  for every  $n$ . By efficiency,  $\pi_*|\psi_n|$  tends uniformly to  $|\phi|$  on  $K$ . Since  $\int_K |\phi| < 1 - \epsilon$ , for  $n$  large enough  $||\psi_n|| = \int_X \pi_*|\psi_n| < 1$  and thus  $\Theta(B_Y) = B_X$ . This is the first part of Theorem 1.1.

We will now prove the contrapositive of the second part. If  $\overline{\Theta(B_Y)}$  meets the unit sphere in  $Q(X)$ , then there is a  $\phi \in Q(X)$  with  $||\phi|| = 1$ , and a sequence  $\psi_n \in Q(Y)$  such that

$$\Theta(\psi_n) = \pi_*(\psi_n) = \phi$$

and  $\|\psi_n\| \rightarrow 1$ . Since  $\pi_*|\psi_n| \geq |\phi|$  and

$$\|\psi_n\| - \|\phi\| = \int_X \pi_*|\psi_n| - |\phi| \rightarrow 0,$$

we have that  $\pi_*|\psi_n|$  tends to  $|\phi|$  in the  $L^1$ -sense. By Cauchy's integral formula, the  $L^1$  norm of an analytic function controls its modulus of continuity, so  $\pi_*|\psi_n| \rightarrow |\phi|$  uniformly on compact subsets of  $X$ . Therefore  $\psi_n$  is an efficient sequence. By Theorem 2.1, the covering is amenable.  $\blacksquare$

### 3 Holomorphic families of measures

This section provides the proof of Theorem 2.1.

**Complex probability measures.** Let  $A$  be a set and let  $L^\infty(A)$  denote the Banach space of bounded, complex-valued functions  $f(a)$  on  $A$  with the norm  $\|f\| = \sup |f(a)|$ .

A *complex probability measure* on  $A$  is a map  $\mu : A \rightarrow \mathbb{C}$  such that  $\|\mu\| = \sum_A |\mu(a)| < \infty$  and  $\sum \mu(a) = 1$ . The measure  $\mu(B)$  of a set  $B \subset A$  is  $\sum_B \mu(a)$ . The measure  $\mu$  determines an element of the dual space  $L^\infty(A)^*$  by

$$\mu(f) = \sum_A \mu(a)f(a).$$

**Amenability.** A *mean* on  $A$  is a complex-linear map

$$m : L^\infty(A) \rightarrow \mathbb{C}$$

such that  $m(f)$  is real if  $f$  is real-valued,  $m(f) \geq 0$  if  $f \geq 0$  and the mean of the constant function  $f = 1$  is one. It follows that  $m$  is continuous, so  $m$  belongs to  $L^\infty(A)^*$ .

**Remark.** Elements of  $L^\infty(A)^*$  can be interpreted as finitely additive measures on  $A$ . From this point of view, a mean is simply a positive finitely additive probability measure.

Let  $G$  be a group acting transitively on the set  $A$ . The action of  $G$  is *amenable* if there exists a  $G$ -invariant mean on  $A$ . Note that amenability is a property of the pair  $(G, H)$  where  $H$  is the stabilizer of a point on  $A$ , since we may identify  $A$  and  $G/H$ .

**Example.** The action of  $G = \mathbb{Z}$  on itself by translation is amenable. However it seems impossible to constructively exhibit a translation invariant

mean that is defined for *all* bounded functions. Thus it is useful to have a more concrete criterion for amenability, and this is provided by the Følner-Rosenblatt condition [Ros]:

An action is amenable if and only if there is a net  $A_\alpha$  of finite subsets of  $A$  such that for every  $g$  in  $G$ ,

$$|gA_\alpha \Delta A_\alpha|/|A_\alpha| \rightarrow 0$$

as  $\alpha \rightarrow \infty$ .

Here  $B\Delta C = (B-C)\cup(C-B)$  denotes the symmetric difference. Frequently the net can be replaced by a sequence. When  $G = A = \mathbb{Z}$  we can take  $A_n = \{0, 1, \dots, n\}$ .

**Remark.** In [Mc1], the proof of Theorem 1.1 employs the Følner-Rosenblatt criterion; the present proof works directly with linear functionals on  $L^\infty(A)$ .

A more complete discussion of amenability can be found in [Mc1], [Gre] and [Pier].

**Amenable coverings.** We now return to the setting of the introduction, and define a *covering*  $\pi : Y \rightarrow X$  to be amenable if the action of  $\pi_1(X)$  on the coset space  $\pi_1(X)/\pi_1(Y)$  is amenable.

**Proof of Theorem 2.1.** Starting with an efficient sequence  $\pi_*(\psi_n) = \phi$ , we will show that the covering  $\pi : Y \rightarrow X$  is amenable.

Let  $E \subset X$  be the analytic subvariety on which  $\phi$  vanishes. To simplify notation, let

$$A(x) = \pi^{-1}(x)$$

denote the fiber of the covering over  $x$ .

We begin by defining, for each  $x$  in  $X - E$ , a sequence of complex probability measures  $\mu_n(x)$  on  $A(x)$ . Let

$$\mu_n(x)(y) = \frac{\psi_n(y)}{\pi^*\phi(x)}.$$

Since  $\mathcal{L}' = \pi^*\mathcal{L}$ ,  $\psi_n$  and  $\pi^*\phi$  are sections of the same line bundle and their ratio is a complex number (finite because  $x$  is not in  $E$ ). Note that

$$\frac{1}{\phi(x)} \sum_{y \in A(x)} \psi_n(y) = \frac{\phi(x)}{\phi(x)} = 1$$

by the definition of pushforward, so these are indeed probability measures. The measures  $\mu_n(x)$  are sections of the flat bundle of Banach spaces  $L^\infty(A(x))^*$ .

**Test functions.** A *test function*  $(f, U)$  is a bounded locally constant function on  $\pi^{-1}(U)$ , where  $U$  is an open subset of  $X - E$ . The function  $f$  can be thought of as a constant section of  $L^\infty(A(x))$ . A family of functionals  $\mu(x) \in L^\infty(A(x))$  determines a map  $\langle \mu, f \rangle: U \rightarrow \mathbb{C}$  by  $x \mapsto \mu(x)(f)$ .

Since  $\pi: Y \rightarrow X$  is a covering map, any bounded function on a fiber of  $\pi$  can be prolonged to a test function. Thus  $\mu(x)$  is determined by its values on test functions.

**Proposition 3.1** 1. *The measures  $\mu_n(x)$  depend holomorphically on  $x$  in  $X - E$ . That is,  $\langle \mu_n, f \rangle$  is holomorphic on  $U$  for any test function  $(f, U)$ .*

2. *For any compact  $K \subset X - E$ ,  $\|\mu_n(z)\| \rightarrow 1$  uniformly on  $K$  as  $n$  tends to infinity.*

**Proof.** Let  $(f, U)$  be a test function. Then

$$\mu_n(x)(f) = \frac{1}{\phi(x)} \sum_{\pi(y)=x} \psi_n(y)f(y).$$

The sum converges uniformly on compact sets by the definition of pushforward, so  $\mu_n(x)$  is holomorphic.

As for the second statement,

$$\|\mu_n(x)\| = \sum_{y \in A(x)} |\mu_n(x)(y)| = \frac{\pi_* |\psi_n|(x)}{|\phi(x)|},$$

and by the definition of efficiency this quantity tends to one uniformly on  $K$  as  $n \rightarrow \infty$ . ■

**Proposition 3.2** *There exists a subnet  $\mu_\alpha$  of  $\mu_n$  which converges weak\*-uniformly on compact sets to a holomorphic functional  $\nu(x) \in L^\infty(A(x))^*$ .*

This means that  $\langle \mu_\alpha, f \rangle$  converges to  $\langle \nu, f \rangle$  uniformly on compact sets for any test function  $(f, U)$ .

**Proof.** Let  $L^\infty(A)_1^*$  denote the unit ball endowed with the weak\* topology. By theorems of Tychonoff and Alaoglu [Roy], the space

$$\prod_X L^\infty(A(x))_1^*$$

is compact. Since  $\|\mu_n\| \rightarrow 1$ , the  $\mu_n$  are locally bounded, so there is a subnet  $\mu_\alpha(x)$  which converges pointwise to  $\nu(x)$ . Now for each test function  $(f, U)$ , and each compact  $K \subset U$ ,  $\langle \mu_\alpha, f \rangle$  is a pointwise convergent net of uniformly bounded holomorphic functions. Since bounded holomorphic functions are equicontinuous, the convergence is uniform and the limit is holomorphic. ■

**Proposition 3.3** *The functional  $\nu$  is a locally constant mean on  $A(x)$ .*

Here locally constant means that  $\langle \nu, f \rangle$  is constant on  $U$  for any test function  $(f, U)$  defined over a connected set  $U$ .

**Proof.** Since  $\mu_\alpha$  is a net of complex probability measures,  $\langle \mu_\alpha, 1 \rangle = 1$ , so the same is true for  $\nu$ .

We now appeal to the following easy fact: if  $a_i$  is a sequence of complex numbers, with  $\sum a_i = 1$  and  $\sum |a_i| = 1 + \epsilon$ , then  $\sum_{\operatorname{Re} a_i < 0} |a_i| \leq \epsilon$  and  $\sum |\operatorname{Im} a_i| < \delta(\epsilon)$  where  $\delta \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Since  $\mu_\alpha$  is a subnet of  $\mu_n$ , we have

$$\pi_* |\mu_\alpha(x)| = \sum_{y \in A(x)} |\mu_\alpha(y)| \rightarrow 1,$$

and we have just noted that  $\sum_{A(x)} \mu_\alpha(y) = 1$ . It follows that the limiting functional  $\nu(x)$  is real and  $\nu(x)(f) \geq 0$  if  $f \geq 0$ . Thus  $\nu$  is a mean.

A real-valued holomorphic function is constant. Since  $\langle \nu, f \rangle$  is holomorphic for any test function  $f$ , and real-valued if  $f$  is real-valued, it follows that  $\nu$  is locally constant. ■

**Completion of the proof of Theorem 2.1.** Let  $x_0$  be a basepoint in  $X - E$ ,  $y_0$  a point in  $Y$  lying over  $x_0$ . The choice of  $y_0$  determines an action of  $G = \pi_1(X, x_0)$  on  $A(x_0)$ , with stabilizer  $H$  of  $y_0$  equal to  $\pi_1(Y, y_0)$ . Thus  $\nu(x_0)$  determines a mean on  $L^\infty(G/H)$ .

We claim this mean is  $G$ -invariant. Since  $E$  has real codimension two, any loop in  $\pi_1(X, x_0)$  has a representative  $\gamma$  which avoids  $E$ . The monodromy of the locally constant family of sets  $A(x)$  around  $\gamma$  is exactly the action of  $\gamma$  on the fiber  $A(x_0) \cong G/H$ . But  $\nu(x)$  is a locally constant functional on  $L^\infty(A(x))$ , so it is invariant under the action of  $\gamma$ .

Thus  $\nu$  is an invariant mean on  $L^\infty(G/H)$  and the covering  $Y \rightarrow X$  is amenable. ■

## 4 A bridge to affine bundles

To conclude, we give an interpretation of the Barrett-Diller proof of the Theta conjecture [BD], which may illuminate the connection of their work with the argument above.

**Definitions.** An *affine automorphism* of  $\mathbb{C}$  is simply a transformation of the form  $z \mapsto az + b$ , where  $a, b \in \mathbb{C}$  and  $a \neq 0$ . A *complex affine line*  $L$  is a space equipped with a bijection  $L \rightarrow \mathbb{C}$  well-defined up to composition with affine automorphisms.

Given a countable bounded set  $A \subset L$  and a complex probability measure  $\mu$  on  $A$ , define the *barycenter*  $\beta$  of  $\mu$  by

$$\beta = \sum_{a \in A} \mu(a)a.$$

Since  $\sum_A \mu(a) = 1$ , the point  $\beta \in L$  is well-defined: it is independent of the particular identification of  $L$  with  $\mathbb{C}$  needed to form the sum.

Unlike the barycenter of a positive measure, the barycenter of a complex measure can lie outside the convex hull of its support. But it is easy to see:

**Proposition 4.1** *If  $\sum_A |\mu(a)| = 1 + \epsilon$ , and  $A$  is contained in a round disk  $D(p, r)$  in  $L$ , then the barycenter of  $\mu$  lies in  $D(p, r(1 + \epsilon))$ .*

Now let  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  denote the Riemann sphere. For any point  $z \in \widehat{\mathbb{C}}$ , there is an automorphism (Möbius transformation)  $\delta : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  such that  $\delta(z) = \infty$ . This  $\delta$  is well-defined up to composition with an affine automorphism. Consequently:

The complement  $L = \widehat{\mathbb{C}} - \{z\}$  of any point  $z \in \widehat{\mathbb{C}}$  carries the structure of an affine line.

For  $\mu$  a complex probability measure on a countable set  $A \subset \widehat{\mathbb{C}}$ , let  $\text{bary}(\mu, z)$  denote the barycenter of  $\mu$  with respect to the natural affine structure on  $\widehat{\mathbb{C}} - \{z\}$ . This point is defined so long as  $\overline{A}$  does not meet  $z$ .

The barycenter is *natural*, in the sense that  $\text{bary}(\delta_*\mu, \delta z) = \delta \text{bary}(\mu, z)$  for any automorphism  $\delta$  of  $\widehat{\mathbb{C}}$ .

**Barycenter approach to the Theta conjecture.** With these preliminaries in place, we now sketch the Barrett-Diller proof of the Theta conjecture.

Let  $\Delta = \{z : |z| < 1\}$  denote the unit disk, and let  $\Sigma = \widehat{\mathbb{C}} - \overline{\Delta}$  denote the complementary disk about infinity.

Let  $X$  be a hyperbolic Riemann surface of finite volume. Then we can find a covering map  $\pi : \Sigma \rightarrow X$  which presents  $X$  as the quotient of the disk  $\Sigma$  by the action of a Fuchsian group  $\Gamma$ .

The group  $\Gamma$  acts on the whole Riemann sphere, and  $\overline{X} = \Delta/\Gamma$  is also a Riemann surface, the ‘‘complex conjugate’’ of  $X$ . More precisely, reflection through the unit circle commutes with the action of  $\Gamma$ , and so it determines an antiholomorphic homeomorphism  $\rho : X \rightarrow \overline{X}$ .

Now suppose  $\|\Theta_{\Delta/X}\| = 1$ . Since the unit ball in  $Q(X)$  is compact, we can find a  $\phi \in Q(X)$  and a sequence  $\psi_n \in Q(\Delta)$  such that  $\Theta_{\Delta/X}(\psi_n) = \phi$  and  $\lim \|\psi_n\| = \|\phi\| = 1$ . From this we will deduce a contradiction.

Let  $\psi = \pi^*\phi$  denote the pullback of  $\phi$  to  $\Sigma$ , and let  $E \subset \Sigma$  be the discrete set of zeros of  $\psi$ .

Pick a point  $p$  in  $\Delta$ , and let  $A$  denote the orbit  $\Gamma p$ . For  $z \in \Sigma - E$ , define a complex probability measure  $\mu_n(z)$  on  $A$  by

$$\mu_n(z)(\gamma p) = \frac{\psi_n(\gamma^{-1}z)}{\psi(z)}.$$

We claim that for any  $\delta \in \Gamma$ , the pushforward  $\delta_*\mu_n(z) = \mu_n(\delta z)$ . Indeed,

$$(\delta_*\mu_n(z))(\gamma p) = \mu_n(z)(\delta^{-1}\gamma p) = \frac{\psi_n(\gamma^{-1}\delta z)}{\psi(z)} = \frac{\psi_n(\gamma^{-1}\delta z)}{\psi(\delta z)} = \mu_n(\delta z)(\gamma p),$$

where we have used the fact that  $\psi$  is  $\Gamma$ -invariant.

Define  $f_n : \Sigma - E \rightarrow \widehat{\mathbb{C}}$  by

$$f_n(z) = \text{bary}(\mu_n(z), z);$$

i.e.  $f_n(z)$  is the barycenter of  $\mu_n(z)$  with respect to  $z$ . It is not hard to verify that  $f_n$  is holomorphic. Moreover, for any  $\delta \in \Gamma$ ,

$$f_n(\delta z) = \text{bary}(\mu_n(\delta z), \delta z) = \text{bary}(\delta_*\mu_n(z), \delta z) = \delta \text{bary}(\mu_n(z), z) = \delta f_n(z)$$

by naturality of the barycenter.

To conclude the proof, use the fact that  $\|\psi_n\| \rightarrow \|\phi\| = 1$  to construct a subsequence  $f_{n_k}$  converging to  $g(z)$  uniformly on compact subsets of  $\Sigma - E$ . By Proposition 4.1 above, one may show that in the limit,  $g(z)$  lies within the convex hull of  $A$  with respect to the affine structure determined by  $z$ . In particular,  $g$  maps  $\Sigma - E$  into the closure of the unit disk  $\Delta$ . Since bounded analytic functions have no isolated singularities,  $g$  can be extended to a map  $\Sigma \rightarrow \overline{\Delta}$ .

The key property of this map is that it inherits equivariance from the  $f_n$ : that is,  $g(\delta z) = \delta g(z)$  for all  $\delta$  in  $\Gamma$ .

It is easy to see that no such  $g$  exists. If  $g$  assumes a value in  $\partial\Delta$  it must be constant. The constant must be a common fixed point for all elements of  $\Gamma$ , which is absurd for a cofinite volume Fuchsian group.

Otherwise  $g$  maps  $\Sigma$  into  $\Delta$ . Composing with the reflection  $\rho$  through the unit circle, we obtain a map  $\rho \circ g : \Sigma \rightarrow \Sigma$  such that  $\rho \circ g(\delta z) = \delta(\rho \circ g(z))$  for  $\delta$  in  $\Gamma$ . This map descends to an antiholomorphic map  $h : X \rightarrow X$  which induces the identity on  $\pi_1(X)$ .

By the Schwarz lemma,  $h$  must be an orientation reversing isometry, since it cannot shrink the length of a closed geodesic. Thus the original map  $g$  is a Möbius transformation exchanging  $\Sigma$  and  $\Delta$ . But since  $g$  commutes with  $\Gamma$ , it preserves the attracting fixed points of hyperbolic elements of  $\Gamma$ ; these are dense in the circle, so  $g$  is the identity. This contradicts the fact that  $g$  maps  $\Sigma$  to  $\Delta$ . ■

**Remark.** It is hard to imagine an analogue of the orientation-reversing map  $h$  for a general covering  $Y \rightarrow X$ . The proof of Theorem 2.1 was obtained by studying the measures  $\mu_n$ .

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