

# The Mandelbrot set is universal

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## Abstract

We show small Mandelbrot sets are dense in the bifurcation locus for any holomorphic family of rational maps.

## 1 Introduction

Fix an integer  $d \geq 2$ , and let  $p_c(z) = z^d + c$ . The *generalized Mandelbrot set*  $M_d \subset \mathbb{C}$  is defined as the set of  $c$  such that the Julia set  $J(p_c)$  is connected. Equivalently,  $c \in M_d$  iff  $p_c^n(0)$  does not tend to infinity as  $n \rightarrow \infty$ . The traditional Mandelbrot set is the quadratic version  $M_2$ .

A *holomorphic family of rational maps over  $X$*  is a holomorphic map

$$f : X \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$$

where  $X$  is a complex manifold and  $\widehat{\mathbb{C}}$  is the Riemann sphere. For each  $t \in X$  the family  $f$  specializes to a rational map  $f_t : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ , denoted  $f_t(z)$ . For convenience we will require that  $X$  is *connected* and that  $\deg(f_t) \geq 2$  for all  $t$ .

The *bifurcation locus*  $B(f) \subset X$  is defined equivalently as the set of  $t$  such that:

1. The number of attracting cycles of  $f_t$  is not locally constant;
2. The period of the attracting cycles of  $f_t$  is locally unbounded; or
3. The Julia set  $J(f_t)$  does not move continuously (in the Hausdorff topology) over any neighborhood of  $t$ .

It is known that  $B(f)$  is a closed, nowhere dense subset of  $X$ ; its complement  $X - B(f)$  is also called the  *$J$ -stable set* [MSS], [Mc2, §4.1].

As a prime example,  $p_c(z) = z^d + c$  is a holomorphic family parameterized by  $c \in \mathbb{C}$ , and its bifurcation locus is  $\partial M_d$ . See Figure 1.

In this paper we show that *every* bifurcation set contains a copy of the boundary of the Mandelbrot set or its degree  $d$  generalization. The Mandelbrot sets  $M_d$  are thus *universal*; they are initial objects in the category

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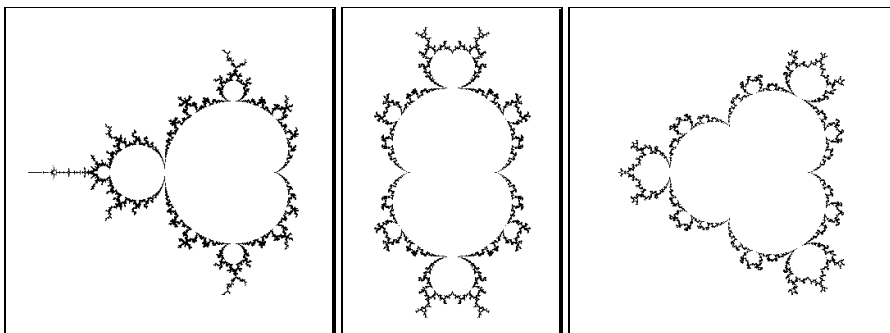


Figure 1. Mandelbrot sets of degrees 2, 3 and 4.

of bifurcations, providing a lower bound on the complexity of  $B(f)$  for all families  $f_t$ .

For simplicity we first treat the case  $X = \Delta = \{t : |t| < 1\}$ .

**Theorem 1.1** *For any holomorphic family of rational maps over the unit disk, the bifurcation locus  $B(f) \subset \Delta$  is either empty or contains the quasi-conformal image of  $\partial M_d$  for some  $d$ .*

The proof (§4) shows that  $B(f)$  contains copies of  $\partial M_d$  with arbitrarily small quasiconformal distortion, and controls the degrees  $d$  that arise. For example we can always find a copy of  $\partial M_d$  with  $d \leq 2^{2^{\deg(f_t)-2}}$ , and generically  $B(f)$  contains a copy of  $\partial M_2$  (see Corollary 4.4). Since the Theorem is local we have:

**Corollary 1.2** *Small Mandelbrot sets are dense in  $B(f)$ .*

There is also a statement in the dynamical plane:

**Theorem 1.3** *Let  $f$  be a holomorphic family of rational maps with bifurcations. Then there is a  $d \geq 2$  such that for any  $c \in M_d$  and  $m > 0$ , the family contains a polynomial-like map  $f_t^n : U \rightarrow V$  hybrid conjugate to  $z^d + c$  with  $\text{mod}(U - V) > m$ .*

**Corollary 1.4** *If  $f$  has bifurcations then for any  $\epsilon > 0$  there exists a  $t$  such that  $f_t(z)$  has a superattracting basin which is a  $(1 + \epsilon)$ -quasidisk.*

**Proof.** The family contains a polynomial-like map  $f_t^n : U \rightarrow V$  hybrid conjugate to  $p_0(z) = z^d$ , a map whose superattracting basin is a round disk. Since  $\text{mod}(V - U)$  can be made arbitrarily large, the conjugacy can be made nearly conformal, and thus  $f_t$  has a superattracting basin which is a  $(1 + \epsilon)$ -quasidisk. ■

For applications to Hausdorff dimension we recall:

**Theorem 1.5 (Shishikura)** *For any  $d \geq 2$ , the Hausdorff dimension of  $\partial M_d$  is two. Moreover  $\text{H. dim}(J(p_c)) = 2$  for a dense  $G_\delta$  of  $c \in \partial M_d$ .*

This result is stated for  $d = 2$  in [Shi2] and [Shi1] but the argument generalizes to  $d \geq 2$ . Quasiconformal maps preserve sets of full dimension [GV], so from Theorems 1.1 and 1.3 we obtain:

**Corollary 1.6** *For any family of rational maps  $f$  over  $\Delta$ , the bifurcation set  $B(f)$  is empty or has Hausdorff dimension two.*

**Corollary 1.7** *If  $f$  has bifurcations, then  $\text{H. dim}(J(f_t)) = 2$  for a dense set of  $t \in B(f)$ .<sup>1</sup>*

For higher-dimensional families one has (§5):

**Corollary 1.8** *For any holomorphic family of rational maps over a complex manifold  $X$ , either  $B(f) = \emptyset$  or  $\text{H. dim}(B(f)) = \text{H. dim}(X) = 2 \dim_{\mathbb{C}} X$ .*

Similar results on Hausdorff dimension were obtained by Tan Lei, under a technical hypothesis on the family  $f$  [Tan].

A family of rational maps  $f$  is *algebraic* if its parameter space  $X$  is a quasi-projective variety (such as  $\mathbb{C}^n$ ) and the coefficients of  $f_t(z)$  are rational functions of  $t$ . For example,  $p_c(z) = z^d + c$  is an algebraic family over  $X = \mathbb{C}$ . Such families almost always contain bifurcations [Mc1]:

**Theorem 1.9** *For any algebraic family of rational maps, either*

1. *The family is trivial ( $f_t$  and  $f_s$  are conformally conjugate for all  $t, s \in X$ ); or*
2. *The family is affine (every  $f_t$  is critically finite and double covered by a torus endomorphism); or*
3. *The family has bifurcations ( $B(f) \neq \emptyset$ ).*

**Corollary 1.10** *With rare exceptions, any algebraic family of rational maps exhibits small Mandelbrot sets in its parameter space.*

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<sup>1</sup>This set of  $t$  can be improved to a dense  $G_\delta$  using Shishikura's idea of hyperbolic dimension.

This Corollary was our original motivation for proving Theorem 1.1.

As another application, for  $t \in \mathbb{C}^{d-1}$  let

$$f_t(z) = z^d + t_1 z^{d-2} + \cdots + t_{d-1}$$

and let

$$\mathcal{C}_d = \{t : J(f_t) \text{ is connected}\}$$

denote the *connectedness locus*. Then we have:

**Corollary 1.11 (Tan Lei)** *The boundary of the connectedness locus has full dimension; that is,  $\text{H. dim}(\partial\mathcal{C}_d) = \text{H. dim}(\mathcal{C}_d) = 2d - 2$ .*

**Proof.** Consider the algebraic family  $g_a(z) = z^d + az^{d-1}$ , which for  $a \neq 0$  has all but one critical point fixed under  $g_a$ . By Theorem 1.9, this family has bifurcations at some  $a \in \mathbb{C}$ . Then there is a neighborhood  $U$  of  $(a, 0, \dots, 0) \in \mathbb{C}^{d-2}$  such that for  $t \in U$  all critical points of  $f_t$  save one lie in an attracting or superattracting basin. If  $t \in B(f) \cap U$ , then the remaining critical point has a bounded forward orbit under  $f_t$ , but under a small perturbation tends to infinity. It follows that  $B(f) \cap U = \partial\mathcal{C}_d \cap U \neq \emptyset$ , and thus  $\dim(\partial\mathcal{C}_d) \geq \dim B(f) = 2d - 2$ . ■

**Remark.** Rees has shown that the bifurcation locus has positive measure in the space of all rational maps of degree  $d$  [Rs]; it would be interesting to know general conditions on a family  $f$  such that  $B(f)$  has positive measure in the parameter space  $X$ .

**Acknowledgements.** I would like to thank Tan Lei for sharing her results which prompted the writing of this note. Special cases of Theorem 1.1 were developed independently by Douady and Hubbard [DH, pp.332-336] and Eckmann and Epstein [EE].

## 2 Families of rational maps

In this section we begin a more formal study of maps with bifurcations.

**Definitions.** A *local bifurcation* is a holomorphic family of rational maps  $f_t(z)$  over the unit disk  $\Delta$ , such that  $0 \in B(f)$ .

The following natural operations can be performed on  $f$  to construct new local bifurcations:

1. *Coordinate change:* replace  $f_t$  by  $m_t \circ f_t \circ m_t^{-1}$ , where  $m : \Delta \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is a holomorphic family of Möbius transformations.
2. *Iteration:* replace  $f_t(z)$  by  $f_t^n(z)$  for a fixed  $n \geq 1$ .

3. *Base change*: replace  $f_t(z)$  by  $f_{\phi(t)}(z)$ , where  $\phi : \Delta \rightarrow \Delta$  is a nonconstant holomorphic map with  $\phi(0) \in B(f)$ .

The first two operations leave the bifurcation locus unchanged, while the last transforms  $B(f)$  to  $\phi^{-1}(B(f))$ .

**Marked critical points.** We will also consider pairs  $(f, c)$  consisting of a local bifurcation and a *marked critical point*; this means  $c : \Delta \rightarrow \widehat{\mathbb{C}}$  is holomorphic and  $f'_t(c_t) = 0$ . The operations above also apply to  $(f, c)$ ; a coordinate change replaces  $c_t$  with  $m_t(c_t)$  and a base change replaces  $c_t$  with  $c_{\phi(t)}$ .

**Misiurewicz points.** A marked critical point  $c$  of  $f$  is *active* if its forward orbit

$$\langle f_t^n(c_t) : n = 1, 2, 3, \dots \rangle$$

fails to form a normal family of functions of  $t$  on any neighborhood of  $t = 0$  in  $\Delta$ . A parameter  $t$  is a *Misiurewicz point* for  $(f, c)$  if the forward orbit of  $c_t$  under  $f_t$  lands on a repelling periodic cycle. If  $t = 0$  is a Misiurewicz point, then either  $c$  is active or  $c_t$  is preperiodic for all  $t$ .

**Proposition 2.1** *If  $c$  is an active critical point, then  $(f, c)$  has a sequence of distinct Misiurewicz points  $t_n \rightarrow 0$ .*

**Proof.** This is a traditional normal families argument. Choose any 3 distinct repelling periodic points  $\{a_0, b_0, c_0\}$  for  $f_0$ , and let  $\{a_t, b_t, c_t\}$  be holomorphic functions parameterizing the corresponding periodic points of  $f_t$  for  $t$  near zero. Since  $\langle f_t^n(c_t) \rangle$  is not a normal family, it cannot avoid these three points, and any parameter  $t$  where  $f_t^n(c_t)$  meets  $a_t, b_t$  or  $c_t$  is a Misiurewicz point. ■

**Ramification.** Next we discuss the existence of univalent inverse branches for a single rational map  $F(z)$ . Let  $d = \deg(F, z)$  denote the local degree of  $F$  at  $z \in \widehat{\mathbb{C}}$ ; we have  $d > 1$  iff  $z$  is a critical point of multiplicity  $(d - 1)$ . We say  $y$  is an *unramified preimage* of  $z$  if for some  $n \geq 0$ ,  $F^n(y) = z$  and  $\deg(F^n, y) = 1$ . We say  $z$  is *unramified* if it has infinitely many unramified preimages. In this case its unramified preimages accumulate on the full Julia set  $J(F)$ .

**Proposition 2.2** *If  $z$  has 5 distinct unramified preimages then it has infinitely many.*

**Proof.** Let  $E$  be the set of all unramified preimages of  $z$ , and let  $C$  be the critical points of  $F$ . Then  $F^{-1}(E) \subset E \cup C$ , so if  $|E|$  is finite then

$$d|E| = \sum_{z \in F^{-1}(E)} 1 + \text{mult}(f', z) \leq |F^{-1}(E)| + 2d - 2 \leq |E| + 4d - 4$$

and therefore  $|E| \leq 4$ . ■

**Corollary 2.3** *Let  $(f, c)$  be a local bifurcation with marked critical point. Then the set of  $t$  such that  $c_t$  is ramified for  $f_t$  is either discrete or the whole disk.*

**Proof.** By the previous Proposition, the ramified parameters are defined by a finite number of analytic equations in  $t$ . ■

**Proposition 2.4** *After a suitable base change, any local bifurcation  $f$  can be provided with an active marked critical point  $c$  such that  $c_0$  is unramified for  $f_0$ .*

**Remark.** It is possible that all the active critical points are ramified at  $t = 0$ . The base change in the Proposition will generally not preserve the central fiber  $f_0$ .

**Proof.** The set  $C = \{(t, z) \in \Delta \times \widehat{\mathbb{C}} : f'_t(z) = 0\}$  is an analytic variety with a proper finite projection to  $\Delta$ . By Puiseux series, after a base change of the form  $\phi(t) = \epsilon t^n$  all the critical points of  $f$  can be marked by holomorphic functions  $\{c_t^1, \dots, c_t^m\}$ . Since  $t = 0$  is in the bifurcation set, by [Mc2, Thm. 4.2], there is an  $i$  such that  $\langle f_t^n(c_t^i) \rangle$  is not a normal family at  $t = 0$ . That is,  $c^i$  is an active critical point.

Next we show  $c^i$  can be chosen so that for generic  $t$  it is disjoint from the forward orbits of all other critical points. If not, there is a  $c^j$  and  $n \geq 1$  such that  $f_t^n(c_t^j) = c_t^i$  for all  $t$ . Then  $c^j$  is also active and we may replace  $c^i$  with  $c^j$ . If the replacement process were to cycle, then  $c^i$  would be a periodic critical point, which is impossible because it is active. Thus we eventually achieve a  $c^i$  which is generically disjoint from the forward orbits of the other critical points.

In particular, there is a  $t$  such that  $c_t^i$  is unramified for  $f_t$ . By Corollary 2.3, the set  $R \subset \Delta$  of parameters where  $c_t^i$  is ramified is discrete. By Proposition 2.1, there are Misiurewicz points  $t_n$  for  $(f, c^i)$  with  $t_n \rightarrow 0$ . Choose  $n$  such that  $t_n \notin R$ , and make a base change moving  $t_n$  to zero; then  $c^i$  is active, and  $c_0^i$  is unramified for  $f_0$ . ■

**Misiurewicz bifurcations.** Let  $(f, c)$  be a local bifurcation with a marked critical point. We say  $(f, c)$  is a *Misiurewicz bifurcation* of degree  $d$  if

- M1.  $f_0(c_0)$  is a repelling fixed-point of  $f_0$ ;
- M2.  $c_0$  is unramified for  $f_0$ ;
- M3.  $f_t(c_t)$  is not a fixed-point of  $f_t$ , for some  $t$ ; and
- M4.  $\deg(f_t, c_t) = d$  for all  $t$  sufficiently small.

**Proposition 2.5** *For any local bifurcation  $(f, c)$  with  $c$  active and  $c_0$  unramified, there is a base change and an  $n > 0$  such that  $(f^n, c)$  is a Misiurewicz bifurcation.*

**Remark.** The delicate point is condition (M4). The danger is that for every Misiurewicz parameter  $t$ , the forward orbit of  $c_t$  might accidentally collide with another critical point before reaching the periodic cycle. We must avoid these collisions to make the degree of  $f_t^n$  at  $c_t$  locally constant.

**Proof.** There are Misiurewicz points  $t_n \rightarrow 0$  for  $(f, c)$ , and  $c_t$  is unramified for all  $t$  near 0, so after a base change and replacing  $f$  with  $f^n$  we can arrange that  $(f, c)$  satisfies conditions (M1), (M2) and (M3).

We can also arrange that  $\deg(f_t, c_t) = d$  for all  $t \neq 0$ . However (M4) may fail because  $\deg(f_t, c_t)$  may jump up at  $t = 0$ . This jump would occur if another critical of  $f_t$  coincides with  $c_t$  at  $t = 0$ .

To rule this out, we make a further perturbation of  $f_0$ . Let  $a_t$  locally parameterize the repelling fixed-point of  $f_t$  such that  $f_0(c_0) = a_0$ . Choose a neighborhood  $U$  of  $a_0$  such that for  $t$  small,  $f_t$  is linearizable on  $U$  and  $U$  is disjoint from the critical points of  $f_t$ . (This is possible since  $f_0'(a_0) \neq 0$ .)

Let  $b_t \in U - \{a_t\}$  be a parameterized repelling periodic point close to  $a_t$ . Then  $b_t$  has preimages in  $U$  accumulating on  $a_t$ . Choose  $s$  near 0 such that  $f_s(c_s)$  hits one of these preimages (such an  $s$  exists by the argument principle and (M3)). For this special parameter,  $c_s$  first maps close to  $a_s$ , then remains in  $U$  until it finally lands on  $b_s$ . Since there are no critical points in  $U$ , we have  $\deg(f_s^i, c) = d$  for all  $i > 0$ .

Making a base change moving  $s$  to  $t = 0$ , we find that  $(f^n, c)$  satisfies (M1-M4) for  $n$  a suitable multiple of the period of  $b_s$ . ■

### 3 The Misiurewicz cascade

In this section we show that when a Misiurewicz point bifurcates, it produces a cascade of polynomial-like maps.

**Definitions.** A *polynomial-like map*  $g : U \rightarrow V$  is a proper, holomorphic map between simply-connected regions with  $\bar{U}$  compact and  $\bar{U} \subset V \subset \mathbb{C}$  [DH]. Its *filled Julia set* is defined by

$$K(g) = \bigcap_1^{\infty} g^{-n}(V);$$

it is the set of points that never escape from  $U$  under forward iteration.

Any polynomial such as  $p_c(z) = z^d + c$  can be restricted to a polynomial-like map  $p_c : U \rightarrow V$  of degree  $d$  with the same filled Julia set. Moreover small analytic perturbations of  $p_c : U \rightarrow V$  are also polynomial-like.

A degree  $d$  Misiurewicz bifurcation  $(f, c)$  gives rise to polynomial-like maps  $f_t^n : B_0 \rightarrow B_n$ , by the following mechanism. For small  $t$ , a small ball  $B_0$  about the critical point  $c_t$  maps to a small ball  $B_1$  close to, but not containing, the fixed-point of  $f_t$ . The iterates  $B_i = f_t^i(B)$  then remain near the fixed-point for a long time, ultimately expanding by a large factor. Finally for suitable  $t$ , as  $B_i$  escapes from the fixed-point it maps back over  $B_0$ , resulting in a degree  $d$  map  $f_t^n : B_0 \rightarrow B_n \supset B_0$ . Since most of the images  $\langle B_i \rangle$  lie in the region where  $f_t$  behaves linearly, the first-return map  $f_t^n : B_0 \rightarrow B_n$  behaves like a polynomial of degree  $d$ .

This scenario leads to a cascade of families of polynomial-like maps, indexed by the return time  $n$ . Here is a precise statement.

**Theorem 3.1** *Let  $(f, c)$  be a degree  $d$  Misiurewicz bifurcation, and fix  $R > 0$ . Then for all  $n \gg 0$ , there is a coordinate change depending on  $n$  such that  $c_t = 0$  and*

$$f_t^n(z) = z^d + \xi + O(\epsilon_n)$$

whenever  $|z|, |\xi| \leq R$ . Here  $t = t_n(1 + \gamma_n \xi)$ ,  $t_n$  and  $\gamma_n$  are nonzero, and  $\gamma_n$ ,  $t_n$  and  $\epsilon_n$  tend to zero as  $n \rightarrow \infty$ .

The constants in  $O(\cdot)$  above depend on  $f$  and  $R$  but not on  $n$ .

The proof yields more explicit information. Let  $\lambda_0 = f'_0(f_0(c_0))$  be the multiplier of the fixed-point on which  $c_0$  lands, and let  $r$  be the multiplicity of intersection of the graph of  $c_t$  and the graph of this fixed-point at  $t = 0$ . Then for  $t = t_n$ , the critical point  $c_t$  is periodic with period  $n$ , and we have:

$$t_n \sim C\lambda_0^{-n/r}, \quad (3.1)$$

$$\gamma_n = C'\lambda_0^{-n/(d-1)}, \quad \text{and} \quad (3.2)$$

$$\epsilon_n = n(|\lambda_0|^{-n/r} + |\lambda_0|^{-n/(d-1)}), \quad (3.3)$$

for certain constants  $C, C'$  depending on  $f$ . Due to the choice of roots, there are  $r$  possibilities for  $t_n$  and  $(d-1)$  for  $\gamma_n$ ; the Theorem is valid for all choices. Finally for  $\xi$  fixed and  $t = t_n(1 + \gamma_n \xi)$ , the map  $f_t^n$  is polynomial-like near  $c_t$  for all  $n \gg 0$ , and in the *original*  $z$ -coordinate its filled Julia set satisfies

$$\text{diam } K(f_t^n) \asymp |\lambda_0|^{-n/(d-1)}.$$

**Notation.** We adopt the usual conventions:  $a_n = O(b_n)$ ,  $a_n \asymp b_n$ ,  $a_n \sim b_n$  and  $n \gg 0$  mean  $|a_n| < C|b_n|$ ,  $(1/C)|b_n| < |a_n| < C|b_n|$ ,  $a_n/b_n \rightarrow 1$  and  $n \geq N$ , where  $C$  and  $N$  are implicit constants.

**Proof.** We will make several constructions that work on a small neighborhood of  $t = 0$ . First, let  $a_t$  parameterize the repelling fixed-point of  $f_t$  such

that  $a_0 = f_0(c_0)$ . Let  $\lambda_t = f'_t(a_t)$  be its multiplier. There is a holomorphically varying coordinate chart  $u = \phi_t(z)$  defined near  $z = a_t$  such that

$$\phi_t \circ f_t \circ \phi_t^{-1}(u) = \lambda_t u \tag{3.4}$$

for  $u$  near 0. We call  $u = \phi_t(z)$  the *linearizing coordinate*; note that  $u = 0$  at  $a_t$ .

We next arrange that  $u = 1$  is an unramified preimage of  $c_t$ . Since  $c_0$  is unramified by (M2), its unramified preimages accumulate on  $a_0$ . Let  $b_0$  be one such preimage, with  $f_0^p(b_0) = c_0$  and  $b_0$  in the domain of  $\phi_0$ . Then  $b_0$  prolongs to a holomorphic function  $b_t$  with  $f_t^p(b_t) = c_t$ . Replacing  $\phi_t$  by  $\phi_t(z)/\phi_t(b_t)$ , we can assume  $u = \phi_t(b_t) = 1$ .

For small  $t$ , the composition  $f_t^p \circ \phi_t^{-1}$  is univalent near  $u = 1$ . By applying a coordinate change  $z \mapsto m_t(z)$ , where  $m_t$  is a Möbius transformation depending on  $t$ , we can arrange that  $c_t = 0$  and that

$$f_t^p \circ \phi_t^{-1}(u) = (u - 1) + O((u - 1)^2) \tag{3.5}$$

on  $B(1, \epsilon)$ .

Since  $\deg(f_t, 0) = d$  for  $t$  near 0 by (M4), we have

$$\phi_t \circ f_t(z) = \sum A_i(t) z^i \tag{3.6}$$

$$= A_0(t) + A_d(0) z^d (1 + O(|z| + |t|)) \tag{3.7}$$

with  $A_d(0) \neq 0$ . Here  $A_0(t) = f_t(0)$  is the  $u$ -coordinate of the critical value. By (M3),  $c_t$  is not pre-fixed for all  $t$ , so there is an  $r > 0$  such that

$$A_0(t) = t^r B(t) \tag{3.8}$$

where  $B(0) \neq 0$ .

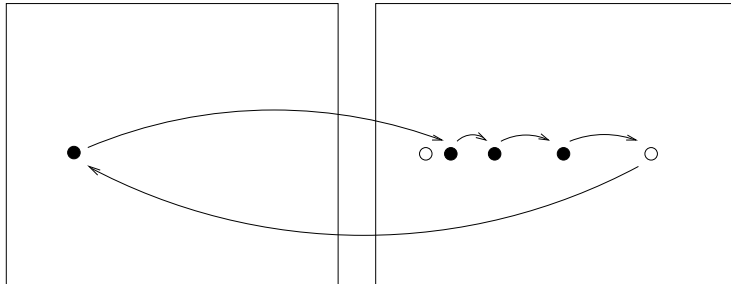


Figure 2. Visiting the repelling fixed-point

Next for  $n \gg 0$  we choose  $t_n$  such that

$$f_t^{1+n+p}(c_t) = c_t \quad \text{when } t = t_n. \tag{3.9}$$

More precisely, for  $t = t_n$  we will arrange that  $c_t$  maps first close to  $a_t$ , then lands after  $n$  iterates on  $b_t$ , and thus returns in  $p$  further iterates to  $c_t$ ; see Figure 2. In the  $u$ -coordinate system,  $f_t$  is linear and  $b_t = 1$ , so the equation  $f_t^{n+1}(c_t) = b_t$  becomes

$$\lambda_t^n A_0(t) = 1 \quad \text{when } t = t_n. \quad (3.10)$$

By the argument principle, for  $n \gg 0$  this equation has a solution  $t_n$  close to any root of the approximation  $\lambda_0^n t^r B(0) = 1$  obtained from (3.8). Moreover

$$t_n \sim B(0)^{-1} \lambda_0^{-n/r}$$

(verifying (3.1)), and  $t_n$  satisfies (3.9) because  $f_t^p(b_t) = c_t$ . (There are actually  $r$  solutions for  $t_n$  for a given  $n$ ; any one of the  $r$  solutions will do.)

We now turn to the estimate of  $f_t^{1+n+p}(z)$  for  $(t, z)$  near  $(t_n, 0)$ . We will assume throughout that  $t = t_n + s$  and that:

$$|z| \text{ and } |s/t_n| \text{ are } O(\Lambda^{-n/(d-1)}) \quad (3.11)$$

where  $\Lambda = |\lambda_0| > 1$ . (To see this is the correct scale at which to work, suppose  $\text{diam}(B) \asymp \text{diam } f_t^{1+n+p}(B)$ , where  $B$  is a ball centered at  $z = 0$ . Then  $\text{diam } f_t(B) \asymp (\text{diam } B)^d$ , and  $f_t^n$  is expanding by a factor of about  $\Lambda^n$ , while  $f_t^p$  is univalent, so we get  $\text{diam } B \asymp \Lambda^n (\text{diam } B)^d$ , or  $\text{diam } B \asymp \Lambda^{-n/(d-1)}$ . Similarly  $|f_t^{1+n+p}(0)| \asymp \Lambda^n (s/t_n) t_n^r \asymp (s/t_n) = O(\text{diam } B)$  when  $s$  is as above.)

It is also convenient to set

$$\tilde{\Lambda} = \min(\Lambda^{1/(d-1)}, \Lambda^{1/r}) > 1,$$

so that we may assert:

$$z \text{ and } t \text{ are } O(\tilde{\Lambda}^{-n}). \quad (3.12)$$

By (3.11) the  $n$  iterates of  $f_t(z)$  lie within the domain of linearization, so by (3.7) we have

$$\phi_t \circ f_t^{1+n}(z) = \lambda_t^n A_0(t) + \lambda_t^n A_0(d) z^d (1 + O(|z| + |t|)).$$

The first term is approximately 1. Indeed,  $\lambda_t^n = \lambda_{t_n}^n (1 + O(ns))$ , so by (3.8) we have

$$\begin{aligned} \lambda_t^n A_0(t) &= \lambda_t^n (t_n + s)^r B(t_n + s) \\ &= \lambda_{t_n}^n (1 + O(ns)) \cdot t_n^r \left(1 + \frac{s}{t_n}\right)^r \cdot B(t_n) (1 + O(s)) \\ &= \lambda_{t_n}^n A_0(t_n) \left(1 + r \frac{s}{t_n} + O((s/t_n)^2) + O(ns)\right) \\ &= 1 + r \frac{s}{t_n} + O((s/t_n)^2) + O(ns) \end{aligned}$$

by (3.10). Similarly,  $\lambda_t^n = \lambda_0^n(1 + O(t))$ , so

$$\begin{aligned} \phi_t \circ f_t^{1+n}(z) - 1 &= \\ \lambda_0^n A_0(d) z^d (1 + O(|z| + |nt|)) + r \frac{s}{t_n} + O((s/t_n)^2) + O(ns) &= \\ \lambda_0^n A_0(d) z^d + r \frac{s}{t_n} + O(n\Lambda^{-n/(d-1)} \tilde{\Lambda}^{-n}), \end{aligned}$$

using (3.11) and (3.12). The expression above as a whole is  $O(\Lambda^{-n/(d-1)})$ , so composing with the univalent map  $f_t^p \circ \phi_t^{-1}$  introduces (by (3.5)) an additional error of size  $O(\Lambda^{-2n/(d-1)})$ , which is already accounted for in the  $O(\cdot)$  above. Thus the expression above also represents  $f_t^{1+n+p}(z)$ .

Finally we make a linear change of coordinates of the form  $z \mapsto \alpha_n z$ , conjugating the expression above to

$$f_t^{1+n+p}(z) = \alpha_n^{1-d} \lambda_0^n A_0(d) z^d + \alpha_n r \frac{s}{t_n} + O(n\alpha_n \Lambda^{-n/(d-1)} \tilde{\Lambda}^{-n}).$$

Setting  $\alpha_n = (\lambda_0^n A_0(d))^{1/(d-1)}$  to normalize the coefficient of  $z^d$ , we have  $|\alpha_n| \asymp \Lambda^{n/(d-1)}$  and thus:

$$\begin{aligned} f_t^{1+n+p}(z) &= z^d + \alpha_n r \frac{s}{t_n} + O(n\tilde{\Lambda}^{-n}) \\ &= z^d + \xi + O(\epsilon_n) \end{aligned}$$

with  $t = t_n(1 + \gamma_n \xi)$ ,  $\gamma_n$  and  $\epsilon_n$  as in (3.2) and (3.3). Notice that if  $|z|$  and  $|\xi|$  are bounded by  $R$  in the expression above, then (3.11) is satisfied in our original coordinates. Reindexing  $n$ , we obtain the Theorem.  $\blacksquare$

## 4 Small Mandelbrot sets

We now show the Misiurewicz cascade leads to small Mandelbrot sets in parameter space. From this we deduce Theorems 1.1 and 1.3 of the Introduction.

**Hybrid conjugacy.** Let  $g_1, g_2$  be polynomial-like maps of the same degree. A *hybrid conjugacy* is a quasiconformal map  $\phi$  between neighborhoods of  $K(g_1)$  and  $K(g_2)$  such that  $\phi \circ g_1 = g_2 \circ \phi$  and  $\bar{\partial}\phi|_{K(g_1)} = 0$ . We say  $g_1$  and  $g_2$  are *hybrid equivalent* if such a conjugacy exists. By a basic result of Douady and Hubbard, every polynomial-like map  $g$  of degree  $d$  is hybrid equivalent to a polynomial of degree  $d$ , unique up to affine conjugacy if  $K(g)$  is connected [DH, Theorem 1].

**Theorem 4.1** *Let  $(f, c)$  be a degree  $d$  Misiurewicz bifurcation. Then the parameter space  $\Delta$  contains quasiconformal copies  $\mathcal{M}_d^n$  of the degree  $d$  Mandelbrot set  $M_d$ , converging to the origin, with  $\partial\mathcal{M}_d^n$  contained in the bifurcation locus  $B(f)$ .*

*More precisely, for all  $n \gg 0$  there are homeomorphisms*

$$\phi_n : M_d \rightarrow \mathcal{M}_d^n \subset \Delta$$

*such that:*

1.  $f_t^n$  is hybrid equivalent to  $z^d + \xi$  whenever  $t = \phi_n(\xi)$ ;
2.  $d(0, \mathcal{M}_d^n) \asymp |\lambda_0|^{-n/r}$ ;
3.  $\text{diam}(\mathcal{M}_d^n)/d(0, \mathcal{M}_d^n) \asymp |\lambda_0|^{-n/(d-1)}$ ;
4.  $\phi_n$  extends to a quasiconformal map of the plane with dilatation bounded by  $1 + O(\epsilon_n)$ ; and
5.  $\psi_n^{-1} \circ \phi_n(\xi) = \xi + O(\epsilon_n)$ , where  $\psi_n(\xi) = t_n(1 + \gamma_n \xi)$ .

The notation is from (3.1) – (3.3).

We begin by recapitulating some ideas from [DH]. Let  $\Delta(R) = \{z : |z| < R\}$ , and let

$$g_\xi(z) = z^d + \xi + h(\xi, z)$$

be a holomorphic family of mappings defined for  $(\xi, z) \in \Delta(R) \times \Delta(R)$ , where  $R > 10$  and  $g'_\xi(0) = 0$ . Let  $\mathcal{M} \subset \Delta(R)$  be the set of  $\xi$  such that the forward orbit  $g_\xi^n(0)$  remains in  $\Delta(R)$  for all  $n > 0$ .

**Lemma 4.2** *There is a  $\delta > 0$  such that if  $\sup |h(\xi, z)| = \epsilon < \delta$  then there is a homeomorphism*

$$\phi : M_d \rightarrow \mathcal{M}$$

*such that for all  $\xi \in M_d$ ,  $g_{\phi(\xi)}$  is hybrid equivalent to  $z^d + \xi$ ,  $|\phi(\xi) - \xi| < O(\epsilon)$ , and  $\phi$  extends to a  $1 + O(\epsilon)$ -quasiconformal map of the plane.*

**Proof.** Let  $p_\xi(z) = z^d + \xi$ . Since  $R > 10$  we have  $M_d \subset \Delta(R)$  and  $K(p_\xi) \subset \Delta(R)$  for all  $\xi \in M_d$ ; indeed these sets have capacity one, so their diameters are bounded by 4 [Ah]. In addition, for  $\xi \in M_d$  the map  $p_\xi : U \rightarrow \Delta(R)$  is polynomial-like, where  $U = p_\xi^{-1}(\Delta(R))$ . By continuity, when  $\sup |h|$  is sufficiently small,  $\mathcal{M}$  is compact and  $g_\xi$  is polynomial-like for all  $\xi \in \mathcal{M}$ .

By results of Douady and Hubbard, we can also choose  $\delta$  small enough that  $|h| < \delta$  implies there is a homeomorphism

$$\phi : M_d \rightarrow \mathcal{M}$$

such that  $g_{\phi(\xi)}$  is hybrid equivalent to  $z^d + \xi$  [DH, Prop. 21].

Now assume  $|h| < \epsilon < \delta$ . For  $t \in \Delta$  let  $\mathcal{M}_t$  denote the parameters where the critical point remains bounded for the family

$$g_{\xi,t} = z^d + \xi + t \frac{\delta}{\epsilon} h(\xi, z),$$

and define  $\phi_t : M_d \rightarrow \mathcal{M}_t$  as above. Then  $\phi_t$  is a family of injections, with  $\phi_0(z) = z$ , and  $\phi_t(\xi)$  is a holomorphic function of  $t$  for every  $\xi$ . (For example this is clear at  $\xi \in \partial M_d$  because Misiurewicz points are dense in  $\partial M_d$ ; for the general case see [DH, Prop. 22].)

By a theorem of Ślodkowski [Sl] (cf. [Dou], [BR]),  $\phi_t(z)$  prolongs to a holomorphic motion of the entire plane, and its complex dilatation  $\mu_t = \bar{\partial}\phi_t/\partial\phi_t$  gives a holomorphic map of the unit disk into the unit ball in  $L^\infty(\widehat{\mathbb{C}})$ . By the Schwarz lemma,  $\|\mu_t\|_\infty \leq |t|$ ; since  $\phi = \phi_{\epsilon/\delta}$ , we obtain a quasiconformal extension of  $\phi$  with dilatation  $1 + O(\epsilon)$ . The bound on  $|\phi(\xi) - \xi|$  similarly results by applying the Schwarz Lemma to the map  $\Delta \rightarrow \widehat{\mathbb{C}} - \{0, 1, \infty\}$  given by  $t \mapsto \phi_t(\xi)$ , once three points have been normalized to remain fixed during the motion. ■

**Proof of Theorem 4.1.** Fix  $R > 10$ . For all  $n \gg 0$ , Theorem 3.1 provides a family of rational maps of the form

$$g_\xi(z) = f_t^n(z) = z^d + \xi + O(\epsilon_n)$$

defined for  $(\xi, z) \in \Delta(R) \times \Delta(R)$ , where  $t = \psi_n(\xi) = t_n(1 + \gamma_n \xi)$ . The preceding Lemma gives homeomorphisms  $\widetilde{\phi}_n : M_d \rightarrow \widetilde{\mathcal{M}}_d^n \subset \Delta(R)$  for all  $n \gg 0$ . Setting  $\phi_n = \psi_n \circ \widetilde{\phi}_n$ , the Theorem results from the Lemma and the bounds (3.1) – (3.3). ■

**Example.** The quadratic family  $(f, c) = (z^2 + t - 2, 0)$  is a Misiurewicz bifurcation of degree  $d = 2$ , with  $\lambda_0 = 2$  and  $r = 1$ . Thus  $M_2$  contains small copies  $\mathcal{M}_2^n$  of itself near  $c = -2$ , with  $d(\mathcal{M}_2^n, -2) \asymp 4^{-n}$  and  $\text{diam } \mathcal{M}_2^n \asymp 16^{-n}$ .

**Consequences.** Assembling the preceding results, we may now prove the Theorems stated in the Introduction. Here is a more precise form of Theorem 1.1:

**Theorem 4.3** *Let  $f$  be a holomorphic family of rational maps over the unit disk with bifurcations. Then there is a nonempty list of degrees*

$$D \subset \{2, 3, \dots, 2^{2 \deg(f_t) - 2}\}$$

*such that for any  $\epsilon > 0$  and  $d \in D$ ,  $B(f)$  contains the image of  $\partial M_d$  under a  $(1 + \epsilon)$ -quasiconformal map.*

*If the critical points of  $f$  are marked  $\{c_t^1, \dots, c_t^m\}$  such that*

$$(i \leq N) \iff c^i \text{ is active and } c_t^i \text{ is unramified for some } t,$$

then we may take

$$D = \{\inf_t \sup_k \deg(f_t^k, c_t^i) : i \leq N\}.$$

**Proof.** Let  $B_0 = B(f)$ . After a base change we can assume that  $f$  is a local bifurcation with critical points marked as above. By Proposition 2.4, there is at least one active, unramified critical point, so  $N \geq 1$ . For any  $i \leq N$ , we can make a base change so  $c_t^i$  is active and unramified; then by Proposition 2.5, a further base change makes  $(f^n, c^i)$  into a degree  $d$  Misiurewicz bifurcation.

Let  $d_i = \inf_t \sup_k \deg(f_t^k, c_t^i)$ . We claim  $d = d_i \leq 2^{2 \deg(f_t) - 2}$ . Indeed,  $\deg(f_t^n, c_t^i)$  assumes its minimum outside a discrete set, and it is equal to  $d$  near  $t = 0$ , so  $d_i \geq d$ . On the other hand,  $c_0^i$  lands on a repelling periodic cycle, so  $\deg(f_0^k, c_0^i) = d$  for all  $k > n$ , and therefore  $d_i \leq d$ . Finally  $d$  is largest if  $c^i$  hits all the other critical points of  $f$  before reaching the repelling cycle; in this case  $d = (p_1 + 1)(p_2 + 1) \cdots (p_m + 1)$  for some partition  $p_1 + p_2 + \cdots + p_m = 2d - 2$ . The product is maximized by the partition  $1 + 1 + \cdots + 1$ , so  $d \leq 2^{2 \deg(f_t) - 2}$ .

By Theorem 4.1, the bifurcation locus  $B(f)$  contains almost conformal copies  $\partial \mathcal{M}_d^n$  of  $\partial M_d$  accumulating at  $t = 0$ , with  $\text{diam}(\mathcal{M}_d^n) \ll d(0, \mathcal{M}_d^n)$ . Letting  $\phi : \Delta \rightarrow \Delta$  denote the composition of all the base-changes occurring so far, we have  $B(f) = \phi^{-1}(B_0)$ . Then  $\phi$  is univalent and nearly linear on  $\mathcal{M}_d^n$  for  $n \gg 0$ , so  $\phi(\partial \mathcal{M}_d^n) \subset B_0$  is a  $(1 + \epsilon)$ -quasiconformal copy of  $\partial M_d$ . ■

Let  $\text{Rat}_d$  be the space of all rational maps of degree  $d$ ; it is a Zariski-open subset of  $\mathbb{P}^{2d+1}$ . We now make precise the statement that a generic family contains a copy of the standard Mandelbrot set.

**Corollary 4.4** *There is a countably union of proper subvarieties  $R \subset \text{Rat}_d$  such that for any local bifurcation, either  $f_t \in R$  for all  $t$ , or  $B(f)$  contains a copy of  $\partial M_2$ .*

**Proof.** On a finite branched cover  $X$  of  $\text{Rat}_d$ , the critical points of  $f \in \text{Rat}_d$  can be marked  $\{c^1(f), \dots, c^{2d-2}(f)\}$ . Clearly  $\deg(f^n, c_i(f)) = 2$  outside a proper subvariety  $V_{n,i}$  of  $X$ . Let  $R$  be the union of the images of these varieties in  $\text{Rat}_d$ , and apply the preceding argument to see  $D = \{2\}$  if some  $f_t \notin R$ . ■

**Proof of Theorem 1.3.** The proof follows the same lines as that of Theorem 4.3; to get  $\text{mod}(V - U)$  large one takes  $R$  large in Theorem 3.1. Thus Theorem 1.3 also holds for all  $d \in D$ . ■

## 5 Hausdorff dimension

In this section we prove Corollary 1.8: for any holomorphic family  $f$  of rational maps over a complex manifold  $X$ , we have  $\text{H. dim}(B(f)) = \text{H. dim}(X)$  if  $B(f) \neq \emptyset$ .

Recall that the *Hausdorff dimension* of a metric space  $X$  is the infimum of the set of  $\delta \geq 0$  such that there exists coverings  $X = \bigcup X_i$  with  $\sum (\text{diam } X_i)^\delta$  arbitrarily small.

**Lemma 5.1** *Let  $Y$  be a metric space,  $X$  a subset of  $Y \times [0, 1]$ . Then*

$$\text{H. dim}(X) \geq 1 + \inf \text{H. dim}(X_t)$$

where  $X_t = \{y : (y, t) \in X\}$ .

Here  $Y \times [0, 1]$  is given the product metric.

**Proof.** Fix  $\delta$  with  $\delta + 1 > \text{H. dim}(X)$ . For any  $n$  there is a covering  $X \subset \bigcup B(y_i, r_i) \times I_i$  with  $|I_i| = r_i$  and  $\sum r_i^{\delta+1} < 4^{-n}$ . Note that

$$X_t \subset \bigcup_{t \in I_i} B(y_i, r_i)$$

and

$$\int_0^1 \sum_{t \in I_i} r_i^\delta dt = \sum r_i^{\delta+1} < 4^{-n}.$$

Let  $E_n$  be the set of  $t$  where the integrand exceeds  $2^{-n}$ ; then  $\sum m(E_n) < \sum 2^{-n} < \infty$ . Thus almost every  $t$  belongs to at most finitely many  $E_n$ , so almost every  $X_t$  admits infinitely many coverings with  $\sum r_i^\delta < 2^{-n} \rightarrow 0$ . Therefore  $\delta \geq \inf \text{H. dim}(X_t)$ , and the Theorem follows.  $\blacksquare$

The Lemma above is related to the product formula

$$\text{H. dim}(X \times Y) \geq \text{H. dim}(X) + \text{H. dim}(Y);$$

see [Fal, Ch. 5] and references therein.

**Proof of Corollary 1.8.** Suppose  $B(f) \neq \emptyset$ . Then there is a  $t_0 \in B(f)$  and a locally parameterized periodic point  $a(t)$  of period  $n$  such that  $a(t)$  changes from attracting to repelling near  $t_0$  [MSS], [Mc2, §4.1]. More formally this means the multiplier  $\lambda(t) = (f^n)'(a(t))$  is not locally constant and  $|\lambda(t_0)| = 1$ .

Choosing local coordinates we can reduce to the case  $X = \Delta^n$  and  $t_0 = 0$ . Let  $\Delta_s = \Delta \times \{s\}$  for  $s \in \Delta^{n-1}$ . For coordinates in general position,  $\lambda(t)$  is nonconstant on  $\Delta_0$ . Shrinking the  $\Delta^{n-1}$  factor, we can also assume  $a(t)$  changes from attracting to repelling in the family  $f|_{\Delta_s}$  for all  $s$ . Then

$$B(f)_s = B(f) \cap \Delta_s \supset B(f|_{\Delta_s}) \neq \emptyset$$

and  $\text{H. dim } B(f|\Delta_s) = 2$  by Corollary 1.6. Applying the Lemma above to  $B(f) \subset \Delta \times \Delta^{n-1}$  we find

$$\text{H. dim}(B(f)) \geq (2n - 2) + \inf_s \text{H. dim } B(f)_s = 2n = \text{H. dim}(X).$$

■

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