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Geometric Topology Tutorial
Solution Set # 3

Problem 1. *Let T_n be an infinite tree with $n \geq 3$ edges incident to each vertex. Given a finite subset V of the vertices of T_n, let ∂V consists of the vertices of T_n that are not themselves in V, but lie on edges incident to V. Show there is a constant $\gamma_n > 0$ such that $|\partial V_n| \geq \gamma_n |V|$ for every V, and determine the best (i.e. largest) possible value of γ_n.*

`Solution:` Suppose that the vertices of V are the vertices of a connected subtree. In this case, it is not hard to compute the size of the boundary: if $|V| = k$ then each vertex is incident to n other vertices and since the subtree has $k-1$, we get $|\partial V| = nk - 2(k-1) = (n-2)k + 2$.

Next, by induction on the number of connected components of the subgraph with vertices V, we can show that $|\partial V| \geq (n-2)|V| + 2$. Indeed, suppose V_1 and V_2 are two components, for which the above inequality holds. Notice that the boundaries of V_1 and V_2 cannot have more than one edge in common, since otherwise their union would have a cycle. Therefore

$$|\partial(V_1 \cup V_2)| \geq |\partial V_1| + |\partial V_2| - 1 \geq (n-2)(|V_1| + |V_2|) + 3 \geq (n-2)|V_1 \cup V_2| + 2.$$

Thus $|\partial V|/|V| \geq (n-2) + 2/k$ and as $k \to \infty$, we obtain $\gamma_n = n - 2$. □

Problem 2. *Prove that the free group F_n on $n \geq 2$ generators is not amenable.*

`Solution:` This problem is a direct consequence of #1, since the Cayley graph of F_n is an infinite tree whose vertices have degree $2n$. □

Problem 3. *Show the group $G = \langle a, b : aba^{-1} = b^2 \rangle$ is isomorphic to the of group affine homeomorphisms of $\mathbb{R}$ generated by $a(x) = 2x$ and $b(x) = x + 1$.*

`Solution:` Let G' be the group of affine homeomorphisms of $\mathbb{R}$ generated by $a(x) = 2x$ and $b(x) = x + 1$. Consider the homomorphism $\phi : G \to G'$ that sends $a \mapsto a(x)$ and $b \mapsto b(x)$. It is well defined, since $(a \circ b \circ a^{-1} \circ b^{-2})(x) = 2(\frac{1}{2}(x-2)+1) = x$. Also ϕ is surjective, since the generators of G' are both in the image of ϕ.

Injectivity is a little more subtle. First, notice that the two relations $ab = b^2 a$, $ba' = a'b^2$, $ab' = (b')^2 a$ and $b'a' = a'(b')^2$ allow us to reduce every word to a unique word $(a')^m b^k a^n$, where $m, n \in \mathbb{Z}$ and k is odd (we first reduce every word to $(a')^m b^k a^n$ and then if k is even, we can reduce the power k to $k/2$ by swapping all the b's with one a and continue by induction, until k is odd. For a word in the kernel of ϕ, its reduction must be also in the kernel. But $\phi(a^{-m} b^k a^n) = f$, where f is the affine transformation, for which

$$f(x) = 2^{-m}(2^n x + k) = 2^{n-m} x + \frac{k}{2^m}.$$

Since this affine transformation must be the identity, then we must have $n = m$ and $k = 0$. Therefore, any word in the kernel of φ can be reduced to $a^{-n}a^n = 1$ by the relations in G, i.e. the kernel is trivial. $\square$

Problem 4. Show that G has exponential growth: the number W_n of distinct elements of G that can be expressed as words in $\langle a, b\rangle$ of length $\leq n$ satisfies $W_n \geq C\alpha^n > 0$ for some $\alpha > 1$.

`Solution (due to Ian Le):` Consider words of only a's and b's of length n, which do not contain consecutive b's (i.e. the b's are separated by a's). For each such word, we can push all the b's in front, using the relation $ab = b^2a$. If we think of the words as binary representations of numbers, we can convince ourselves that two different words have different presentations as b^Na^M. Using the homomorphism from the previous problem, we get that if $b^Na^M = b^{N'}a^{M'}$ then $2^Mx+N = 2^{M'}+N'$, i.e. $M = M'$ and $N = N'$. Therefore, distinct such words give rise to distinct elements of the group G. We can prove easily by induction that the number of such words is precisely the n-th Fibonacci number F_n. Therefore, $\#W_n \geq F_n \sim \alpha^n$, where $\alpha = \dfrac{1+\sqrt{5}}{2}$. $\square$

Problem 5. *Prove that G is amenable.*

Problem 6. *Show the free group on 3 generators is a subgroup of the free group on 2 generators. That is, find explicit words $\{w_1, w_2, w_3\}$ in $\{a, b\}$ that generate a free group $\langle w_1, w_2, w_3\rangle$ inside $\langle a, b\rangle$, and prove that they do so. (Hint: you can use the reduced words* or *covering spaces to prove the (w_i) have no relations.)*

`Solutions:` Consider the subgroup $G = \langle a, b^2, bab^{-1}\rangle$. The covering space that this group corresponds to has fundamental group F_3 (this can be seen by choosing a maximal tree) and therefore, there are no relations among the generators. $\square$

Problem 7. Prove the free group on 3 generators is not isomorphic to the free group on 2 generators.

`Solution:` The problem follows from the simple observation that if two groups are isomorphic, then their abelianizations are also isomorphic. But $F_2/[F_2, F_2] \simeq \mathbb{Z}/2$ and $F_3/[F_3, F_3] \simeq \mathbb{Z}/3$, so the two groups cannot be isomorphic. $\square$

Problem 8. *Find generators for the kernel of the map $\langle a, b\rangle \to \langle a, b : a^2, b^3, aba'b'\rangle$.*

`Solution (due to A. Levine):` Consider the canonical homomorphism $\phi : \langle a, b\rangle \to G$. First of all, note that G is abelian and since it is generated by elements of order 2 and 3, it must be isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/3 \simeq \mathbb{Z}/6$. This means that $F_2/\ker(\phi) \simeq \mathbb{Z}/6$. Next, draw the Cayley graph of G, which is a covering space of the bouquet of two circles and its deck group. First, since the graph is path-connected, any deck transformation is uniquely determined by where a single point is sent, so there must be at most 6 deck transformations. But the move ab

is a deck transformation of order 6, so it follows that the deck group is isomorphic to $\mathbb{Z}/6$. It follows in particular that the cover is normal, i.e. $\pi_1(X) \lhd F_2$ and thus $F_2/\pi_1(X) \simeq \mathbb{Z}/6$, which implies that $\ker(\phi) \simeq \pi_1(X)$. To get the generators, we use the algorithm given in 2.1.7 in Stillwell, i.e. we choose a spanning tree for the kernel and for each edge not in T, there is a generator ϵ_i. In particular, we get generators $\epsilon_1 = a^2, \epsilon_2 = abab', \epsilon_3 = bab'a', \epsilon_4 = ab^3a', \epsilon_5 = b^3, \epsilon_6 = ab'ab, \epsilon_7 = b'aba'$. These relations are among the generators of $\pi_1(X)$. $\square$

Problem 9. *Prove that $\langle a, b : a^2 = b^2 \rangle$ contains a subgroup isomorphic to $\mathbb{Z}^2$. (Note: it is not enough just to find commuting elements x and y, you must also show that $x^i y^j = 1$ iff $(i, j) = (0, 0)$.)*

`Solution: (due to A. Levine)` Let $G = \langle a, b : a^2 = b^2 \rangle$. Then by the Seifert-Van Kampen theorem, G can be realized as the fundamental group of a manifold obtained by a square with edges labeled consequently as a, a, b', b'. Using cut-and-paste moves, we can show that this manifold is in fact the Klein bottle. But the torus T is a two-sheeted covering of the Klein bottle (take two identical Klein bottles and glue them along their edges - be careful which one). Therefore, G contains a subgroup that is isomorphic to $\pi_1(T) = \mathbb{Z}^2$.

Remark: There is another approach - one can consider the subgroup of G, generated by ab and ba and prove that it is isomorphic to $\langle x, y : xy = yx \rangle$.