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Geometric Topology Tutorial
Solution Set # 2

Problem 1. Let $A : \mathbb{Z}^a \rightarrow \mathbb{Z}^b$ be a group homomorphism, specified by a $b \times a$ matrix A_{ij} , and let $G_A = \mathbb{Z}^b/A(\mathbb{Z}^a)$. Describe a sequence of elementary moves on matrices, such that G_A is isomorphic to G_B iff A and B are related by a sequence of your moves.

Solution: A lot of the people realized the connection of this problem with presentations of modules over rings (something, which is discussed in details M. Artin's *Algebra*). The helpful statement, which solves the problem is that if A is a presentation matrix for a module M , then the following matrices A' present the same module M :

1. $A' = QAP^{-1}$, where $Q \in GL_m(R)$ and $P \in GL_n(R)$;
2. A' is obtain by deleting a column of zeros;
3. The j -th column of A is e_i , and A' is obtained from A by deleting the i -th row and j -th column.

So we apply the proposition for $R = \mathbb{Z}$ to show that if A and B are related to each other by sequence of moves (note that we use only type 1 move, since A and B must have the same size), then the corresponding \mathbb{Z} -modules are isomorphic. Conversely, if $\mathbb{Z}^b/A\mathbb{Z}^a \simeq \mathbb{Z}^b/B\mathbb{Z}^a$. This means that the presentations of the two groups are related by a finite series of Tietze moves. But it's not hard to show that each such move correspond to a move of type 1 from the above list.

Remark: The type 1 move above is precisely a compact way of expressing integer matrix moves, namely swapping rows and columns and adding a multiple of a row (resp. column) to another row (resp. column). \square

Problem 2. Let $G = \{g_1, \dots, g_n : r_1, \dots, r_m\}$. Let A_{ij} be the total exponent with respect to which g_i occurs in the relation r_j . Using Tietze moves, show that $n - \text{rank}(A_{ij})$ is an invariant of G . Can you describe this invariant without using a presentation for G ?

Solution: The first part of the problem is pretty straightforward - we check explicitly the effect of each of the Tietze moves on the matrix of total exponents. \square

Problem 3. Which graphs are covered by the complete graphs K_5 and $K_{3,3}$ (the utility graph and the pentagram)?

Solution: Note that if the graph H is covered by the graph G , then $|V(H)|$ is a divisor of $|V(G)|$. This means that if $G = K_5$, then $|V(H)| = 1$, or 5. If $|V(H)| = 1$

we get for H the figure-eight graph and we show that it is covered by K_5 . In the second case, we get $H = K_5$.

For the case of $G = K_{3,3}$, H must have either 1, 2, 3 or 6 vertices. we can eliminate the possibilities 1 and 3, since the number of edges coming out of each vertex must be odd, since the degree of each edge in $K_{3,3}$ is odd. In the case $|V(H)| = 2$, we get the *theta graph* and if $|V(H)| = 6$, then we get $K_{3,3}$ itself. \square

Problem 4. Let $H \subset G$ be groups with $[G : H] < \infty$. Prove there is a normal subgroup $N \subset H$ with $[G : N] < \infty$. (Hint: consider the action of G on G/H .) Conclude that if $Y \rightarrow X$ is a finite covering, then there is a finite cover $Z \rightarrow Y$ such that $Z \rightarrow X$ is Galois.

Solution: Consider the subgroup $N = \bigcap_{g \in G} gHg^{-1}$. It is easy to check that $N \triangleleft G$ is a normal subgroup. To show that $[G : N] < \infty$, let G act on G/H by left multiplication. The action corresponds to a homomorphism $\phi : G \rightarrow \text{Perm}(G/H) \cong S_n$, where $n = [G : H]$. The key observation is that $N = \ker(\phi)$. Then $G/N \cong G/\ker(\phi) \cong \text{im}(\phi)$, which is finite.

Suppose $Y \rightarrow X$ is a finite covering. Then $\pi_1(Y) \subset \pi_1(X)$. Hence, there exists a normal subgroup $N \subset \pi_1(Y)$ such that N is a normal subset of $\pi_1(X)$ that is of finite index. N is the fundamental group of a finite covering $Z \rightarrow X$ and this covering is Galois, since N is normal. \square

Problem 5. Let $A_4 \subset S_4$ be the alternating group. Pick generators (a, b) for A_4 with $a^2 = b^3 = \text{id}$, and draw the Cayley graph of A_4 with respect to these generators. Finally give a presentation for A_4 in terms of your generators.

Solution: Let $A_4 \subset S_4$ be the alternating group. Let $a = (12)(34)$ and let $b = (123)$. It's not hard to see that a and b generate all element of A_4 . From the Cayley graph, we notice that $(ab)^3 = 1$. Then $\langle a, b : a^2, b^3, (ab)^3 \rangle$ is a presentation for A_4 . \square

Problem 6. Up to conjugacy, the group $G = A_4$ has unique subgroups H_n of orders $n = 1, 2, 3, 4$ and 12. Let $\phi : \{a, b\} \rightarrow A_4$ be the map from the free group to A_4 given by your generating set, and let $H'_n = \phi^{-1}(H_n) \subset \langle a, b \rangle$. Draw the covering space X_n of the bouquet of 2 circles corresponding to each subgroup $H'_n \subset \langle a, b \rangle$. (Hint: X_1 is the Cayley graph of A_4 , and X_i covers X_j when i divides j .)

Solution (due to A. Levine): Consider the same presentation as in the previous problem, i.e. $G = \langle a, b : a^2, b^3, (ab)^3 \rangle$, where $a = (12)(34)$ and $b = (123)$. Then $H_1 = \{1\}$, $H_2 = \{1, a\}$, $H_3 = \{1, b, b^2\}$ and $H_4 = V = \{1, a, bab^{-1}, b^{-1}ab\}$. The inverse images of H_n therefore correspond to covering spaces X_n of the bouquet of two circles, whose fundamental group is $\langle a, b \rangle$.

The inverse image of $\phi^{-1}(H_1) = \langle a^2, b^3, (ab)^3 \rangle$. The inverse images of the larger subgroups of A_4 are larger, so they have more generators. In particular, $\phi^{-1}(H_n)$ contains the relations that are needed to kill the elements of A_4 that are not in H_n .

So it follows that $\phi^{-1}(H_2) = \langle a^2, b \rangle$, $\phi^{-1}(H_3) = \langle a, b^3 \rangle$, $\phi^{-1}(H_4) = \langle a^2, b^2, (ab)^2 \rangle$. \square

Problem 7. Let $H \subset \langle a, b \rangle$ be the subgroup generated by nontrivial elements x, y of the free group on 2 generators. Invent an algorithm to determine if H is isomorphic to \mathbb{Z} or not.

Solution: The key idea is to test whether x, y commute. If x, y do not commute, then H is not commutative, so in particular it is not isomorphic to \mathbb{Z} . Suppose x and y commute. Since H is a subgroup of a free group, then it is free. But the only commutative free group is \mathbb{Z} , hence $H \cong \mathbb{Z}$ in this case. \square

Problem 8. Show there is a (geographical) map on the Möbius strip that requires more than 4 colors to make sure bordering countries have different hues.

Solution: Most of the people used a partition of the Möbius band example. We can partition the band into six regions, as shown in the picture, so that the Four-Color Map theorem doesn't hold. \square

Problem 9. The real projective plane, \mathbb{RP}^2 , is obtained by identifying the gluing together a 2-disk and a Möbius band along their edges. Prove that $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$, and that the universal cover of \mathbb{RP}^2 is isomorphic to S^2 . Show how \mathbb{RP}^2 can be identified with the space of 1-dimensional subspaces of \mathbb{R}^3 .

Solution: (due to A. Levine) Let S^2 be the 2-sphere embedded in \mathbb{R}^3 . We define a map from S^2 to \mathbb{RP}^2 as follows: for (x, y, z) in the upper hemisphere ($z \geq 0$), map $(x, y, z) \mapsto (x, y)$, and then compose this map with the quotient map $D \rightarrow \mathbb{RP}^2$. For the points on the lower hemisphere, map $(x, y, z) \mapsto (-x, -y)$. It is clear that this gives a well-defined, continuous map $S^2 \rightarrow \mathbb{RP}^2$. The inverse image of a sufficiently small open ball around $p \in \mathbb{RP}^2$ is two disjoint open balls on the surface of S^2 , including points, whose inverse images lie in the xy -plane. Thus, S^2 is a covering space of \mathbb{RP}^2 and the index $[\pi_1(\mathbb{RP}^2) : p_*(\pi_1(S^2))] = 2$, which means that $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2$.