Homework 5
Geometric Topology
Math 99r – Harvard University
Due Monday, 27 October 2003

1. What is the orientable surface obtained from a regular $2n$-gon by gluing side i to side $i+n$? What non-orientable surface do you obtain if you reverse the directions of all the gluings?

2. Let r be a word in the generators of the free group $F_n = \langle g_1, \dots, g_n \rangle$, and let $K(r)$ be the 2-complex naturally associated to the presentation $G = \langle g_1, \dots, g_n \, : \, r \rangle$ of the 1-relator group G.

   When is $K(r)$ a closed surface? A surface with boundary?

3. Let $G = \pi_1(N_g) = \langle x_0, \dots, x_h \, : \, x_0^2 \cdots x_g^2 \rangle$, and let $H = \pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \, : \, [a_1, b_1] \cdots [a_g, b_g] \rangle$. Show explicitly that there is a homomorphism $\phi : G \to \mathbb{Z}/2$ such that $\mathrm{Ker}(\phi) \cong H$. (This comes from the fact that Σ_g is a double-cover of N_g.)

4. Let X be a closed, orientable, triangulated surface. Let G be a finite group of homeomorphisms of X, acting linearly on each triangle and preserving orientation.

   Show that the space X/G is also an orientable surface, and that the quotient map $p : X \to X/G$ is a branched covering.

5. Show that if $gx = y$, then the stabilizers H_x and H_y of $x, y \in X$ are conjugate subgroups of G.

6. For each $z \in X/G$, let $m(z) = |H_x|$, where x is chosen so that $p(x) = z$. Give a formula relating $\chi(X)$, $\chi(X/G)$, $|G|$, and the numbers $m(z)$.

7. Show that if X has genus two, then $|G| \leq 84$.

8. Find a tiling of a surface of genus two into 16 triangles, such that the group of orientation-preserving symmetries of this tiling satisfies $|G| = 48$. (Hint: consider the octahedron).