Homework 3
Geometric Topology
Math 99r – Harvard University
Due Tuesday, 14 October 2003

1. Let T_n be an infinite tree with $n \geq 3$ edges incident to each vertex. Given a finite subset V of the vertices of T_n, let ∂V consists of the vertices of T_n that are not themselves in V, but lie on edges incident to V.

   Show there is a constant $\gamma_n > 0$ such that $|\partial V_n| \geq \gamma_n |V|$ for every V, and determine the best (i.e. largest) possible value of γ_n.

2. Prove that the free group F_n on $n \geq 2$ generators is not amenable.

3. Show the group $G = \langle a, b \ : \ aba^{-1} = b^2 \rangle$ is isomorphic to the of group affine homeomorphisms of $\mathbb{R}$ generated by $a(x) = 2x$ and $b(x) = x + 1$.

4. Show that G has exponential growth: the number W_n of distinct elements of G that can be expressed as words in $\langle a, b \rangle$ of length $\leq n$ satisfies $W_n \geq C\alpha^n > 0$ for some $\alpha > 1$.

5. Prove that G is amenable.

6. Show the free group on 3 generators is a subgroup of the free group on 2 generators. That is, find explicit words $\{w_1, w_2, w_3\}$ in $\{a, b\}$ that generate a free group $\langle w_1, w_2, w_3 \rangle$ inside $\langle a, b \rangle$, and prove that they do so. (Hint: you can use the reduced words *or* covering spaces to prove the (w_i) have no relations.)

7. Prove the free group on 3 generators is not isomorphic to the free group on 2 generators.

8. Find generators for the kernel of the map $\langle a, b \rangle \rightarrow \langle a, b \ : \ a^2, b^3, aba'b' \rangle$.

9. Prove that $\langle a, b \ : \ a^2 = b^2 \rangle$ contains a subgroup isomorphic to $\mathbb{Z}^2$. (Note: it is not enough just to find commuting elements x and y, you must also show that $x^i y^j = 1$ iff $(i, j) = (0, 0)$.)