Homework 1
Geometric Topology
Math 99r – Harvard University
Due Monday, 29 September 2003

The problems below refer to the knots tabulated in the Appendix of Adams' *The Knot Book*, (p.279).

1. Using the Jordan separation theorem, prove the utility graph does not embed in the plane.

2. Read paragraph 2 on p.21 on Stillwell carefully, and apply it to see how then orientation (P_0, P_1, P_2) of a triangle induces an orientation of its edges. Does the result look right? How can you fix it so it makes sense?

3. Draw the Cayley graphs of the groups with presentations $\langle a, b \,:\, a^2, b^2 \rangle$ and $\langle c, d \,:\, c^2, (cd)^2 \rangle$. Then use Tietze moves to prove these groups are isomorphic.

4. Describe a knotted closed loop that can be traced by walking along the stairways and halls of the Science Center, without crossing your path.

5. Show how to switch one crossing of 5_2 so the result K is still knotted. What knot do you get? Use Reidemeister moves to justify your answer.

6. Let N be your student ID number (or any other random number with 9 digits). Draw the knot 9_{18}, and label the crossings $1, 2, \ldots, 9$ from top to bottom. Make a new knot projection K by switching the ith crossing of 9_{18} whenever the ith digit of N is odd. (For example, if $N = 617495277$, then you would switch crossings 2, 3, 5, 6, 8 and 9.)

   Now simplify K as much as possible, and locate an equivalent knot L in the knot tables. Show K and L are the same knot by describing Reidemeister moves that transform K to L.

   Immersed loops. A smooth map $f : S^1 \to \mathbb{R}^2$ is an *immersion* if $f'(x) \neq 0$ for all x. Two immersions f_0 and f_1 are *regularly homotopic* if they can be connected by a continuous family of immersions f_t, $t \in [0, 1]$.

   The image $f(S^1) = L \subset \mathbb{R}^2$ of an immersion in general position is like an (oriented) knot projection where you forget about the over/under data at the crossings. It is a loop with transverse double points.

   Two immersed loops L_0 and L_1 in general position are regularly homotopic iff one can be transformed into the other using Reidemeister moves II and II but not I.

7. Prove that the figure eight (an immersed loop with one crossing) is not regularly homotopic to the unloop.

8. Find an invariant $n(L) \in \mathbb{Z}$ of oriented immersed loops that doesn't change under Reidemeister moves II and III. Using this, show there are infinitely many different regular homotopy classes of immersed loops.

9. Formulate conjectures about the classification of immersed loops in $\mathbb{R}^2$ and S^2 (the 2-sphere) up to regular homotopy.

10. *(Harder) Prove one of your conjectures.