Problem 1

Prove that the sheaf of divisors on a compact Riemann surface satisfies $H^k(X, \text{Div}) = 0$ for $k \geq 1$.

Solution

Since $X$ is compact, finite covers are cofinal. Suppose $\{U_i\}_{i=1}^n$ form a finite cover. Suppose we have a $k$-cocycle $c = (c_I)$ for this cover. Around each point $x \in X$ we can find and open neighborhood $V_x$ with the property that $V_x$ has compact closure and the closure is entirely inside some $U_i$. By compactness, finitely many of those $V_x$ will cover $x$. This gives us a refinement of $U_i$ given by $V_j$. The cocycle $c$ maps to a cocycle for the cover $\{V_j\}$, and where component of $c$ will now have finite support (since the intersections $V_I$ have compact closure inside $U_I$). Now we can refine $\{V_j\}$. Since there are only finitely many of the $V_I$, the (image of) cocycle $c$ will involve only finitely many points in $X$. Thus we can refine $\{V_j\}$ further to a cover $\{W_k\}$ such that each point occurring in the support of $c$ appears in at most one of the $W_k$. But then the cocycle $c$ maps to a cocycle whose only non-zero components are $\text{Div}(W_{k-1})$. Such a cocycle is obviously coboundarous to 0, so $c$ is cobounarous to 0 in a refinement of $\{U_i\}$.

Problem 2

Show that the natural map $H^1(X, \mathcal{O}^*) \longrightarrow H^1(X, \mathcal{M}^*)$ is surjective.

Solution

We have the short exact sequence of sheaves

\[ 0 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{M}^* \longrightarrow \text{Div} \longrightarrow 0 \]

giving rise to

\[ H^1(X, \mathcal{O}^*) \longrightarrow H^1(X, \mathcal{M}^*) \longrightarrow H^1(X, \text{Div}) = 0 \]

where the equality on the right is by the previous problem.
Problem 3

Let \( q(x) \) be a monic polynomial of degree \( 2g + 2 \) with simple zeros, \( g \geq 2 \), and let \( X \) be the hyperelliptic curve \( y^2 = q(x) \). Let \( P_0, P_\infty \in X \) be points with \( y(P_0) = 0 \) and \( y(P_\infty) = \infty \).

(i) Compute \( h^0(nP_0) \) and \( h^0(nP_\infty) \) for all \( n \in \mathbb{Z} \).

(ii) Find a divisor \( D = P - Q \) on \( X \) which is not principal, such that \( 2D \) is principal.

(iii) Find a divisor \( D \) on \( X \) such that \( 2D \) is a canonical divisor. Compute \( h^0(D) \) for your example.

Solution

(i) The point \( P_0 \) is a Weierstrass point, and at the point \( P_0 \) the number of sections grows like half the genus up until \( n = 2g - 1 \), while the point \( P_\infty \) is not a Weierstrass point:

\[
\begin{array}{c|cccccccc}
 n & 0 & 1 & 2 & 3 & 4 & \cdots & 2g-3 & 2g-2 & 2g-1 & >2g-1 \\
 h^0(nP_0) & 0 & 1 & 2 & 3 & \cdots & g-1 & g & g & n-g+1 \\
\end{array}
\]

\[
\begin{array}{c|cccccccc}
 n & 0 & 1 & \cdots & g & g+1 & g+2 & \cdots & 2g-2 & 2g-1 & >2g-1 \\
 h^0(nP_\infty) & 0 & 1 & \cdots & 1 & 2 & 3 & \cdots & g-1 & g & n-g+1 \\
\end{array}
\]

We prove this first for \( P_0 \). The function \( x - x(P_0) \) vanishes to order 2 at \( P_0 \) and nowhere else. So, \( (x - x(P_0))^{-1} \in [2P_0] \) and more generally, \( (x - x(P_0))^{-k} \in [2kP_0] \). These functions are clearly linearly independent, and Riemann-Roch implies that as \( k \) varies between 0 and \( g - 1 \), the functions \( (x - x(P_0))^{-k} \) give a basis for \( [(2g - 2)P_0] \). Since their valuations at \( P_0 \) are all distinct, so they account for all sections of \( nP_0 \) with \( n \leq 2g - 2 \).

Since the degree of \( q \) is even, \( P_\infty \) is one of two points lying over \( \infty \). To prove that \( P_\infty \) is not a Weierstrass point, we only need to show that \( h^0(gP_\infty) = 1 \). By Riemann-Roch, this is equivalent to \( h^0(K - gP_\infty) = 0 \), in other words that no holomorphic one-form vanishes to order \( g \) at \( P_\infty \). But the holomorphic one-forms on \( X \) are given by \( p(x)dx/y \) with \( \deg(p) \leq g-1 \) and such a form vanishes at \( P_\infty \) to order \( g - 1 - \deg(p) < g \).

(ii) Let \( P_0 \) and \( P'_0 \) be two branch points of the map \( x \), and set \( D = P_0 - P'_0 \). Then \( 2D = (x - x(P_0)/(x - x(P'_0))) \), but \( D \) is not principle because if \( \langle f \rangle = D \) then \( \langle f \rangle \) would be a degree 1 map to \( \mathbb{P}^1 \).

(iii) The divisor of \( (x - P_0)^{g-1}dx/y \) is \( (2g - 2)P_0 \), so we can take \( D = (g-1)P_0 \). By the table above, \( h^0(D) = 1 + [(g-1)/2] \).
Problem 4

Let \( X \) be the compact Riemann surface defined by \( y^2 = 1 - x^6 \), and let \( P = (0,1) \in X \) in the coordinates \((x,y)\).

1. Find the least \( n > 0 \) such that \( h^0(nP) > 1 \).

2. For this value of \( n \), find an explicit rational function \( f(x,y) \) such that \( f \in H^0(X,\mathcal{O}_{nP}) \).

3. Compute the principal part of this function,

\[
f(x) = \frac{a_n}{x^n} + \cdots + \frac{a_1}{x} + O(1)
\]

using the local coordinate \( x \) at \( P \).

4. Verify that \( \text{Res}_P(f\omega) = 0 \) for all \( \omega \in \Omega(X) \).

Solution

1. \( X \) has genus 2, so \( h^0(3P) \geq 3 - 2 + 1 > 1 \). On the other hand, if \( h^0(2P) > 0 \) then Riemann-Roch says \( h^1(K - 2P) > 0 \), thus there exists a form \( \omega \in \Omega(X) \) which vanish at \( P \) to order 2.

   The space \( \Omega(X) \) has a basis \( \frac{dx}{y}, \frac{dx}{y} \) which has order of vanishing 0, 1 at \( P \). Thus no linear combination of them can vanish to order 2 at \( P \), hence \( h^0(2P) = 2 \). Hence the least \( n \) is \( n = 3 \).

2. An explicit function is given by \( f = \frac{y+1}{x^3} \). Let \( P' = (0,-1) \), and \( \infty, \infty' \) the two points at infinity. Then \( \text{div}(x) = P + P' - \infty - \infty' \), while \( \text{div}(y+1) = 6P' - 3\infty - 3\infty' \). This gives \( \text{div}\left(\frac{y+1}{x^3}\right) = 3P' - 3P \).

3. We have \( y + 1 = 1 + (1 - x^6)^{1/2} = 1 - \frac{1}{2}x^6 + O(x^{12}) \). Hence \( \frac{y+1}{x^3} = \frac{2}{x^3} + O(1) \).

4. In terms of the basis \( \frac{dx}{y}, \frac{x dx}{y} \) and the expansion in the previous part, we see that \( f\omega \) has Laurent tail involving only \( \frac{1}{x^3} \) and \( \frac{1}{x^2} \), hence \( \text{Res}_P(f\omega) = 0 \).

Problem 5

Let \( D \) be a divisor on a compact Riemann surface \( X \). Show that every element of \( H^1(X,\mathcal{O}_D) \) can be represented by Mittag-Leffler data, i.e. by a cocycle of the form \( g_{ij} = f_i - f_j \in \mathcal{O}_D(U_{ij}) \) where \( f_i \in \mathcal{M}(U_i) \). (You may use the isomorphism \( H^1(X,\mathcal{O}_D)^* \cong \Omega_{-D}(X) \).)
Solution

Any covering has a refinement $\mathcal{U} = \{U_i\}$ with the property that each point with negative coefficient in $D$ is contained in only one set $U_i$. Therefore, any element of $H^1(X, \mathcal{O}_D)$ can be expressed as a Čech cocycle $(g_{ij})$ with each function $g_{ij}$ holomorphic on $U_i \cap U_j$. It follows from the solution to the Mittag-Leffler problem that the cocycle $(g_{ij})$ can be represented by Mittag-Leffler data.

Problem 6

A canonical divisor $K \subset \mathbb{P}^n$ is the formal sum of hypersurfaces determined by the zeros and poles of a meromorphic canonical form $\omega = \omega(z) dz_1 \wedge \cdots \wedge dz_n$. Give an example of a canonical divisor on $\mathbb{P}^2$. What is the degree of a canonical divisor $K$ on $\mathbb{P}^n$? (I.e. for what value of $d$ is $dH$ a canonical divisor, where $H$ is a hyperplane?)

Solution

Let $Z_0, \ldots, Z_n$ be homogeneous coordinates on $\mathbb{P}^n$. In the affine chart $z_i = Z_i/Z_0$, set $\omega = (dz_1/z_1) \wedge \cdots \wedge (dz_n/z_n)$. Clearly, $\omega$ has a simple pole along each of the $n$ coordinate hyperplanes. To find the behavior of $\omega$ at infinity, consider the chart $w_i = Z_i/Z_n; i = 0, \ldots, n-1$. Then

$$\omega = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_{n-1}}{z_{n-1}} \wedge \frac{dz_n}{z_n} = \frac{d(w_1/w_0)}{w_1/w_0} \wedge \cdots \wedge \frac{d(w_{n-1})}{w_{n-1}} \wedge \frac{d(1/w_0)}{1/w_0} = \pm \frac{dw_1}{w_1} \wedge \cdots \wedge \frac{dw_{n-1}}{w_{n-1}} \wedge \frac{dw_0}{w_0}$$

so we see that $\omega$ has a simple pole on the hyperplane at infinity as well. Therefore, $\omega$ has $n + 1$ simple poles, so it has degree $-(n + 1)$. 