Problem 1

Let $\rho = \rho(z)|dz|$ be a smooth conformal metric on a Riemann surface $X$.

1. Show that the local formula

\[ R(\rho) = \Delta \log \rho \, dx \, dy \]

gives rise to a globally well-defined 2-form on $X$, called the Ricci form of $\rho$. (Here $z = x + iy$ is a local analytic coordinate on $X$; you must verify that $R(\rho)$ does not depend on the choice of coordinates.)

2. Compute the curvature of $\rho$, given locally by

\[ K(\rho) = -\rho^{-2} \Delta \log \rho, \]

for $\rho = |dz|$ on $\mathbb{C}$, for $\rho = |dz|/\text{Im}z$ on $\mathbb{H}$, and for $\rho = 2|dz|/(1 + |z|^2)$ on $\hat{\mathbb{C}}$.

3. Show that if $K(\rho) = 0$, then in suitable local coordinates we have $\rho = |dz|$.

Solution

1. We have the formula $R(\rho) = \partial \bar{\partial} \rho$. Under a change of coordinates $z \mapsto w = f(z)$, the local function $\rho$ changes to $(\rho \circ f^{-1})(w)|f'(z)|$. But this $\partial \bar{\partial}|f'(z)| = 0$ since $|f'(z)|$ is harmonic, hence the 2-form in the new coordinate is still $\partial \bar{\partial} \rho$.

2. For $\rho = |dz|$, we have $\rho(z) = 1$, so $\log \rho = 0$ and $K(\rho) = 0$.

For $\rho = |dz|/\text{Im}z$ on $\mathbb{H}$, we have $\rho(z) = 1/\text{Im}z$, and

\[ K(\rho) = -y^2 \Delta \log(y) = -1 \]

For $\rho = 2|dz|/(1 + |z|^2)$, we have using polar coordinates $(r, \theta)$ on $\mathbb{C}$

\[
K(\rho) = -\frac{(1+r^2)^2}{4} \Delta \log(\frac{2}{1+r^2}) \\
= -\frac{(1+r^2)^2}{4} \left( \frac{1}{r^2} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} + \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial}{\partial r}) \right) \log(1+r^2) \\
= 1.
\]
3. If \( K(\rho) = 0 \) then \( \log \rho \) is harmonic and real-valued, so locally we can find a holomorphic function \( f \) such that \( \text{Re} f = \log \rho \). Thus putting \( F \) to be a local antiderivative of \( e^f \), we have \( |F'(z)| = |e^f| = \rho(z) \). Since \( F'(z) \neq 0 \), \( F \) gives a local change of coordinate, and with respect to which the metric becomes

\[
\rho(z)|dz| = |F'(z)||dz| = |d(F(z))|,
\]

**Problem 2**

Let \( \mathcal{H}^1(X) \) be the space of complex harmonic forms on a compact Riemann surface \( X \). Show that

\[
\langle \alpha, \beta \rangle = \frac{i}{2} \int_X \alpha \wedge \overline{\beta}
\]

defines a Hermitian inner product on \( \mathcal{H}^1(X) \) of signature \((g, g)\), where \( g = \dim \Omega(X) \).

**Solution**

The inner product is clearly sesquilinear, and

\[
\langle \beta, \alpha \rangle = \frac{i}{2} \int_X \beta \wedge \overline{\alpha}
\]

\[
= \frac{-i}{2} \int_X \overline{\alpha} \wedge \beta
\]

\[
= \frac{i}{2} \int_X \overline{\alpha} \wedge \beta
\]

\[
= \frac{i}{2} \int_X \alpha \wedge \beta
\]

\[
= \overline{\langle \alpha, \beta \rangle}
\]

Next we show that the inner product is positive definite on holomorphic one-forms, and negative definite on anti-holomorphic one-forms. If \( \omega \in \Omega(X) \) is a non-zero holomorphic one-form, we can write it locally as \( f(z)dz \). Then

\[
\frac{i}{2} \omega \wedge \overline{\omega} = \frac{i}{2} f(z)dz \wedge \overline{f(z)dz}
\]

\[
= \frac{i}{2} |f(z)|^2(dx + idy) \wedge (dx - idy)
\]

\[
= \frac{i}{2} |f(z)|^2(-2idx \wedge dy)
\]

\[
= |f(z)|^2dx \wedge dy
\]

Since the integrand is (almost everywhere) a positive multiple of volume form on \( X \), the inner product \( \langle \omega, \omega \rangle \) is positive. Similarly \( \langle \overline{\omega}, \overline{\omega} \rangle \) is negative. Since \( \mathcal{H}^1 = \Omega(X) \oplus \overline{\Omega(X)} \) and each has dimension \( g \), the Hermitian inner product has signature \((g, g)\).
Problem 3

Give a finite Leray covering of $S^2$ for the sheaf $\mathbb{Z}$. You may assume that a contractible open satisfies $H^i(U, \mathbb{Z}) = 0$ for $i > 0$.

Solution

Put the round metric on $S^2$, normalized so that the diameter of the sphere is 1. Around each point $P$ on $S^2$, the ball of radius $1/4$ (in the round metric) has the property that it is geodesically convex: Any two points in it has a unique shortest geodesic path joining them, which furthermore lies entirely in it. This property is clearly stable under intersection, and implies contractibility, as one can contract along geodesic rays emanating from a chosen point. Since $S^2$ is open, we can find a finite cover by such geodesic balls, and any finite intersection of them is geodesically convex, hence contractible.

Problem 4

Let $X = \mathbb{C}$ with the Zariski topology ($U$ is open iff $U$ is empty or $U = \mathbb{C} - E$ with $E$ finite). In the latter case, define:

$$\mathcal{F}(U) = \{ f \in \mathbb{C}(z) : f \text{ has no poles on } U \};$$

and set $\mathcal{F}(\emptyset) = (0)$. Show that $\mathcal{F}$ is a sheaf on $X$.

Solution

The natural inclusion maps make $\mathcal{F}$ into a pre-sheaf, so we just need to check the two sheaf axioms. First, a rational function $f$ on an open set $U$ is uniquely determined by its restriction to any smaller open, which proves locality. Second, if we are given a finite collection of sets $U_i$ and functions $f_i \in \mathcal{F}(U_i)$ that agree on the pairwise intersections, then by the first statement all of the $f_i$ determine the same rational function on $X$ (call it $f$), and furthermore $f$ can have no poles on any of the sets $U_i$, so $f \in \mathcal{F}(\bigcup_i U_i)$. By construction the function $f$ restricts to $f_i$ on each set $U_i$, so we’ve proved the gluing axiom.

Problem 5

Let $E$ be an abstract set with a finite cover, $E = \bigcup_i^n E_i$, such that $\bigcap E_i = \emptyset$. Show there are functions $\rho_i : E \to \{0, 1\}$ such that $\rho_i(x) = 0$ on $E_i$ and $\sum_i \rho_i(x) = 1$ for all $x$. 

Solution

Let $\rho_i$ to be the characteristic function of the set

$$(E_i)^c - \bigcup_{j<i}(E_j)^c \cap (E_i)^c.$$ 

Problem 6

Consider a finite covering of $X$ by Zariski open sets $U_i = X - E_i$ with $E_i$ finite. Let $V = \bigcap U_i$. Construct a family of linear maps

$$R_i : F(V) \to \mathbb{C}(z)$$

such that $R_i(f)$ has no poles in $E_i$ and $f = \sum R_i(f)$ for all $f \in F(V)$. (Hint: use partial fractions.)

Solution

For any point $w \in \mathbb{C}$ and any $f \in \mathbb{C}(z)$, let $P_w f$ be the principal part of $f$ at $w$. By this, I mean if we expand $f$ as a power series about $w$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - w)^n$$

then

$$P_w f(z) = \sum_{n=-\infty}^{-1} a_n(z - w)^n.$$ 

Let $P_\infty f = f - \sum_{w \in \mathbb{C}} P_w f$, where we write the integral over $\mathbb{C}$ as a sum to emphasize that only finitely many terms are nonzero. By construction $P_\infty f$ is holomorphic on all of $\mathbb{C}$. Also note that $P_w$ is linear for all $w \in \mathbb{C} \cup \infty$, in the sense that $P_w(f + g) = P_w f + P_w g$. The functions $P_w(f)$ has poles only at $w$.

By the previous problem, applied to the sets $E_i$ with empty intersection, we can find functions $\{v_i\}$ functions on $E = \bigcup E_i$ such that $\sum_i v_i(w) = 1$ for all $w \in E$ and $v_i = 0$ on $E_i$. Define

$$R_i(f) = \sum_{w \in E} \rho_i(w) P_w(f)$$

Let’s check that these $R_i$ work. Since $P_w$ is linear, so is $R_i$. The sum defining $R_i(f)$ only involves terms with $\rho_i(w) = 1$, for which we must have $w \notin E_i$, hence $R_i(f)$ has all poles outside $E_i$. Finally,

$$\sum_i R_i(f) = \sum_i \sum_{w \in E} \rho_i(w) R_w(f) = \sum_w P_w(f) = f.$$
**Problem 7**

Prove $H^1(X, \mathcal{F}) = 0$. (Hint: set $f_i = \sum_j R_j(g_{ij})$).

**Solution**

It’s easy to see that $X$ is compact with respect to the cofinite topology, i.e. any open cover has a finite refinement (in fact a subcover). We’ve shown that the map on $H^1$ induced from a refinement is injective, so we need only show that $H^1(X, \mathcal{U}, \mathcal{F}) = 0$ for any finite open cover $\mathcal{U}$.

Let $\mathcal{U} = \{U_i\}$ be a finite open cover of $X$, and choose maps $R_i : \mathcal{F}(\bigcap U_i) \to \mathbb{C}(z)$ as in problem 6. If $(g_{ij})$ is any 1-cocycle with respect to the cover $\mathcal{U}$, define a 0-cochain $(f_i)$ by

$$ f_i = \sum_j R_j(g_{ij}) $$

Since $g_{ij}$ has poles only on $E_i \cup E_j$ and applying $R_j$ kills all poles on $E_j$, $f_i$ has poles only on $E_i$, which shows that $(f_i)$ is indeed a well-defined 0-cochain. Moreover,

$$ \delta(f_i)_{jk} = f_j - f_k $$

$$ = \sum_l R_l(g_{jl}) - \sum_l R_l(g_{kl}) $$

$$ = \sum_l R_l(g_{jl} - g_{kl}) $$

$$ = \sum_l R_l(g_{jk}) $$

$$ = g_{jk} $$

so $(g_{ij})$ is a coboundary. Hence, $H^1(X, \mathcal{F}) = 0$. 