Throughout $X$ is a compact Riemann surface.

**Problem 1**

Consider the Fermat quartic, defined by $X^4 + Y^4 + Z^4 = 0$. It can be built from 12 regular Euclidean octagons by gluing them together so that 3 meet at every vertex. In terms of $(X, Y, Z)$, where are the centers of these octagons? Where are their vertices?

**Solution**

Put $\zeta = e^{2\pi i/8}$.

The centers of the octagon correspond to the 12 points that are fixed points under an automorphism of order 8. An automorphism of order 8 is $[X : Y : Z] \mapsto [Y : iX : Z]$, whose fixed points are $[1 : \zeta : 0]$ and $[1 : \zeta^5 : 0]$. The remaining points are obtained by the orbits of these two under permutation action by $S_3$ on the homogenous coordinates.

The vertices correspond to the 32 fixed points of an automorphism of order 3. One such automorphism is $[X : Y : Z] \mapsto [Z : X : Y]$, whose fixed points are $[1 : \zeta_3 : \zeta_3^2]$ where $\zeta_3$ is any third root of unity. The orbit of these three points under the automorphism of order 8 we wrote down above are the 32 points corresponding to the vertices.

**Problem 2**

Let $S$ be the oriented topological surface obtained by identifying opposite edges of a regular $4g$-gon $P$ in $\mathbb{R}^2$. Then the edges of $\partial P$, taken in order, determine cycles $C_1, \ldots, C_{4g}$ in $H_1(S, \mathbb{Z})$. Compute the intersection pairing $C_i \cdot C_j$, and prove directly that the matrix $J_{ij} = (C_i \cdot C_j)$ with $1 \leq i, j \leq 2g$ has determinant $\pm 1$.

**Solution**

We’ll do the case $g = 2$ The generalization to $g > 2$ is straightforward. We’ll label the edges by the pattern on the left; then the diagram on the right shows the vertex of the octagon:
Problem 3

The intersection matrix can be read off from the diagram at right:

\[
I = \begin{bmatrix}
0 & 1 & -1 & 1 \\
-1 & 0 & 1 & -1 \\
1 & -1 & 0 & 1 \\
-1 & 1 & -1 & 0
\end{bmatrix}
\]

The following matrix is the inverse:

\[
I^{-1} = \begin{bmatrix}
0 & -1 & -1 & -1 \\
1 & 0 & -1 & -1 \\
1 & 1 & 0 & -1 \\
1 & 1 & 1 & 0
\end{bmatrix}
\]

For instance, here’s how you might convince yourself. The intersection matrix is invariant under rotating the octagon, which is equivalent to saying that \(TI = IT\) where \(T\) is the matrix

\[
\begin{bmatrix}
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1
\end{bmatrix}
\]

that corresponds to a rotation of the octagon by 1/8. Now observe that the matrix we called \(I^{-1}\) has the form \([v Tv T^2v T^3v]\). Then once you check that \(Iv = e_1\), it follows that \(IT^3v = T^3Iv = e_j\), so \(II^{-1}\) really is the identity. Since \(I\) is an integral matrix with integral inverse, its determinant is \(\pm 1\).

**Problem 3**

Let \(X\) be a compact Riemann surface of genus \(g\). Use the Riemann-Roch theorem to show directly that the map

\[
\varphi : X^g \rightarrow \text{Pic}_g(X)
\]

is surjective. Here \(\text{Pic}_g(X)\) is the group of linear equivalence classes of divisors of degree \(g\) on \(X\), and \(\varphi(P_1, \ldots, P_g) = [\sum P_i]\). When is \(\dim \varphi^{-1}(\sum^g P_i) > 0\)?
Solution

Let $D$ be any divisor of degree $g$. Riemann-Roch gives

$$h^0(D) = 1 - g + g + h^0(K - D) \geq 1.$$  

So, we can find a nonzero meromorphic function $f$ such that $(f) + D$. Hence, $D$ is linearly equivalent to a sum of $g$ points with multiplicity, i.e. $D$ is in the image of $X^g$.

The dimension of the fiber over $D$ is just the dimension of the space of effective divisors linearly equivalent to $D$, which is equal to $h^0(D) - 1$. Hence, the fiber has positive dimension whenever $\dim h^0(D) > 1$, or equivalently whenever $h^1(K - D) > 0$. These are by definition the special divisors.

Problem 4

Describe the Jacobian of the curve $X$ defined by $y^2 = x(x^4 - 1)$ as a principally polarized complex torus. That is, write $\text{Jac}(X) = \mathbb{C}^2/\Lambda$ for an explicit lattice $\Lambda$, and give the symplectic form on $\Lambda \cong H_1(X, \mathbb{Z})$. Then describe the action of the automorphism $T(x, y) = (ix, \zeta y)$ on $\text{Jac}(X)$, where $\zeta = e^{2\pi i/8}$. In particular, describe the action of $T$ on $H_1(X, \mathbb{Z})$ and show it preserves the symplectic form.

Solution

$X$ is isomorphic to the Riemann surface obtained by gluing the regular octagon as in problem 2, with the automorphism $T$ acting by rotation by one eighth. In problem 2, we wrote down the symplectic form $I$ and the action of $T$ with respect to the basis $\{C_1, C_2, C_3, C_4\}$ of $H_1(X, \mathbb{Z})$. Using that $TC_i = C_{i+1}$ for $1 \leq i \leq 3$ and $TC_4 = -C_1$ and the computation of intersection numbers in problem 2, we see that $T$ preserves the intersection form.

Choosing the basis for $\Omega(X)$ to be $\omega_1 = \frac{dx}{y}$, $\omega_2 = x \frac{dx}{y}$, we see that $T^*\omega_1 = \frac{dx}{\zeta y} = \zeta \omega_1$ and $T^*\omega_2 = \zeta^3 \omega_2$.

Now we have $\int_{TC_i} \omega = \int_C T^* \omega$. Put $\int_{C_1} \omega_1 = a$, $\int_{C_1} \omega_2 = b$, the matrix of periods is given by

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \zeta \\ b \zeta^3 \end{pmatrix}, \begin{pmatrix} a \zeta^2 \\ b \zeta^6 \end{pmatrix}, \begin{pmatrix} a \zeta^3 \\ b \zeta \end{pmatrix}$$

Hence we must have $a \neq 0$, $b \neq 0$, and we could normalize $\omega_1, \omega_2$ so that $a = b = 1$.

Problem 5

Let $G \cong \mathbb{Z}^6$ be the subgroup of $\text{Div}(X)$ generated by the Weierstrass points $P_1, \ldots, P_6$ on a Riemann surface of genus 2.

1. Show that $5P_1 - \sum_{i=2}^6 P_i$ is a principal divisor.

2. Find generators for the subgroup of all principal divisors $H \subset G$. 
3. Describe the image of $G$ in Pic($X$).

Solution

1. One can choose an isomorphism of $X$ with a hyperelliptic curve given by $y^2 = f(x)$ where $\deg f = 5$. The Weierstrass points are the ramification points of the double cover $x : X \to \mathbb{P}^1$. We can choose $f$ so that the zeroes of $f$ correspond to $P_2, \ldots, P_6$ and the unique point at infinity is $P_1$. Then the function $y$ has zeroes at $P_2, \ldots, P_6$ and pole at only $P_1$, hence $\text{div}(y) = 5P_1 - \sum_{i=2}^6 P_i$.

2. We will show that $2P_i - 2P_j$ and $5P_1 - \sum_{i=2}^6 P_i$ generate $H$. We already know that they are principal divisors. Let $G_0$ denote the subgroup of $G$ consisting of elements of degree 0. The above elements generate a subgroup of index 16 in $G_0$. We have a homomorphism $\varphi : G_0 \to \text{Jac}(X)$ which factors through $\text{Jac}(X)[2] \cong \mathbb{F}_4^2$, since $2P_i \sim K$ is canonical. The kernel of $\varphi$ is exactly $H$.

We claim that $\varphi$ is surjective. Indeed, it suffices to see that the 16 elements $P_i - P_1$ ($1 \leq i \leq 6$), $P_i + P_j - 2P_1$ ($2 \leq i < j \leq 5$) are pairwise not linearly equivalent. Indeed any linear equivalence among those elements will give rise to $P_i + P_j \sim P_k + P_l$, which can only happen when when either $\{i, j\} = \{k, l\}$ or $i = j$ and $l = k$, due to the uniqueness of the hyperelliptic involution. Thus $\varphi$ is surjective and $H$ has index 16 in $G_0$, hence it coincides with the subgroup generated by $2P_i - 2P_j$ and $5P_1 - \sum_{i=2}^6 P_i$.

3. By what we have shown above, the image of $G_0$ is $\text{Jac}(X)[2]$, and thus the image of $G$ in Pic($X$) is an extension of $\mathbb{Z}$ by $(\mathbb{Z}/2\mathbb{Z})^4$, so it is $(\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{Z}$.

Problem 6

Recall that $\Omega(X)$ carries a natural inner product, $\langle \alpha, \beta \rangle = (i/2) \int_X \alpha \wedge \overline{\beta}$.

1. Given an orthonormal basis $(\omega_i)$ for $\Omega(X)$, we define the Bergman metric on $X$ by

$$\rho(z)^2|dz|^2 = \sum |\omega_i(z)|^2|dz|^2.$$  

Prove that $\rho$ is independent of the choice of basis.

2. Prove that we can identify $\text{Jac}(X)$ with $\mathbb{C}^g/\Lambda$ in such a way that the Abel-Jacobi map $X \to \text{Jac}(X)$ is an isometry from the Bergman metric on $X$ to the Euclidean metric on $\mathbb{C}^g/\Lambda$.

Solution

1. The Bergman metric has the following interpretation: A vector $v \in T_{X,x}$ is a linear functional on $\Omega_{X,x}$ and hence on $\Omega(X)$ via the evaluation $\Omega(X) \to \Omega_{X,x}$. The Bergman metric assigns to $v$ the norm as a functional on the normed space $\Omega(X)$. Since this description does not involve choosing orthonormal bases for $\Omega(X)$ it is obviously invariant under change of such orthonormal basis.
2. Using an orthonormal basis \( \omega_i \) and fixing a base point \( O \), we can write the Abel-Jacobi map as

\[
AJ_Q : P \mapsto (\int_0^P \omega_i),
\]

which identifies \( \text{Jac} = \mathbb{C}^g / \Lambda \). If \( v \in T_{X,P} \) is a tangent vector, the above map sends it to the vector \( (\omega_i(P)(v)) \), thus the Bergman norm of \( v \) agrees with the standard Euclidean norm on \( \mathbb{C}^g \), thought of as the tangent space of \( \text{Jac}(X) \) at \( AJ_Q(P) \). This shows that \( AJ_Q \) is an isometry.