Throughout $X$ is a compact Riemann surface.

**Problem 1**

Let $f : X \to \mathbb{P}^2$ be a holomorphic map from a compact Riemann surface to the projective plane whose image is not contained in a line. Show that $f(X)$ is an algebraic variety, i.e. show there is a homogeneous polynomial $F(Z_0, Z_1, Z_2)$ such that $f(X)$ is the zero locus of $F$. (Hint: use the fact that $\mathcal{M}(X)$ is a finite extension of $\mathbb{C}(z)$, where $z = Z_1/Z_0$.)

**Solution**

The assumption that $f(X)$ is not contained in a line implies that the function $z = Z_1/Z_0$ is a nonconstant meromorphic function on $X$. Since $X$ is a compact Riemann surface, its field of functions $\mathcal{M}(X)$ is a finite extension of $\mathbb{C}(z)$. Let $w = Z_2/Z_0$. Since $w$ lies in a finite extension of $\mathbb{C}(z)$, there is a polynomial $F(t) \in \mathbb{C}(z)[t]$ such that $F(w) = 0$. After clearing denominators, we may take $F(t)$ to be an element of $\mathbb{C}[z,t]$, and then after homogenizing, we may write $F(z,w) = F(Z_0, Z_1, Z_2)$. We still have that $F(Z_0, Z_1, Z_2)$ vanishes identically on $f(X)$.

The next step is to choose $F$ so that $f(X)$ is the entire zero locus of $F$. Note that a function of degree $d$ on $\mathbb{P}^2$ restricts to a section of a line bundle $\mathcal{O}(d)|_X$. Since $F$ vanishes identically on $X$, it restricts to the zero section. So if $F$ factors as a product of irreducible polynomials $F_i$ of degrees $d_i$, then one of the polynomials $F_i$ must vanish identically on $X$ since otherwise $F$ could vanish at at most $\deg F = \sum d_i$ points. Replacing $F$ by some $F_i$ if necessary, we may take $F$ to be irreducible.

Let $C$ be the vanishing locus of $F$. Then $C$ is an irreducible, possibly singular, plane curve. Let $\tilde{C}$ be a smooth resolution of $C$, it is a connected compact Riemann surface. Away from the singular points of $C$, the map $f$ lifts to a map $\tilde{f} : X^* \to \tilde{C}$, and since $\tilde{C}$ is a compact Riemann surface, this map must extend to a surjection $\tilde{f} : X \to \tilde{C}$. Since the resolution map $\pi : \tilde{C} \to C$ is also a surjection and since by continuity $\pi \circ \tilde{f} = f$, it follows that $C$ is exactly equal to $f(X)$. 
Problem 2

Let $\varphi : X \rightarrow \mathbb{P}^{g-1}$ be the canonical map. Show there is a representation $\rho : \text{Aut}(X) \rightarrow \text{Aut}(\mathbb{P}^{g-1})$ such that $\varphi(g \cdot x) = \rho(g) \cdot \varphi(x)$ for all $x \in X$ and $g \in \text{Aut}(X)$.

Solution

The codomain of the canonical map can be more naturally written as $\mathbb{P}H^0(K)^\ast$. If we identify a point in $\mathbb{P}H^0(K)^\ast$ with its kernel in $H^0(K)$, then the map $\varphi_K$ is given by $\varphi_K(x) = \{\omega : \omega(x) = 0\}$.

There is also a natural representation $\rho^*$ of $\text{Aut}(X)$ on $H^0(K)$ given by pullback: $\rho^*(g) \cdot \omega = (g^{-1})^* \omega$. The inverse makes this a left action. Let $\rho$ be the projectivization of the dual of this representation: if $H$ is a hyperplane in $H^0(K)$, then $\rho(g) \cdot H = \{(g^{-1})^* \omega : \omega \in H\}$.

Having written everything invariantly, it now follows immediately that

\[
\varphi(g \cdot x) = \{\omega : \omega(gx) = 0\} \\
= \{\omega : g^* \omega(x) = 0\} \\
= \{(g^{-1})^* \omega : \omega(x) = 0\} \\
= \rho(g) \cdot \{\omega : \omega(x) = 0\} \\
= \rho(g) \cdot \varphi(x).
\]

Problem 3

An involution on a complex manifold $Z$ is an element $f$ of order in $\text{Aut}(Z)$ is an element of order 2 in $\text{Aut}(Z)$.

1. Prove that the fixed-point set of any involution on $\mathbb{P}^2$ contains a line.

2. Show that every involution on a non-hyperelliptic Riemann surface $X$ of genus 3 has a fixed point.

3. Give an example of a polynomial $p(x)$ of degree 8 such that the hyperelliptic Riemann surface $X$ defined by $y^2 = p(x)$ admits a fixed-point free involution.

4. Let $X$ be a Riemann surface of even genus. Show that every involution on $X$ has a fixed point.

Solution

1. The automorphism group of $\mathbb{P}^2$ is $\text{PGL}_3(\mathbb{C}) = \text{GL}_3(\mathbb{C})/\mathbb{C}^\times$. Suppose $f$ is an automorphism of order 2, represented by a matrix $A$, then $A$ is not a scalar matrix but $A^2 = \lambda$ is. Thus $A$ is diagonalizable and the eigenvalues are $\pm\sqrt{\lambda}$. Thus there is exactly one eigenvalue whose eigenspace has dimension 2. This subspace correspond to a line in $\mathbb{P}^2$ that is fixed by $f$. 

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2. Since $X$ is a genus 3 non-hyperelliptic curve, the canonical map is an embedding $X \hookrightarrow \mathbb{P}(\Omega(X)^*)$. If $f$ is an involution on $X$, by problem 2 it extends to an involution on $\mathbb{P}^2$, which therefore has a fixed line by the previous part. This line must intersect the image of $X$ in the canonical map, hence $f$ has a fixed point on $X$.

3. We take $p(x) = x^8 + 1$. The curve $y^2 = x^8 + 1$ has an involution $(x, y) \mapsto (-x, -y)$. It has no fixed point on the affine part, since a fixed point must have $x = y = 0$. On the other hand, the 2 points at infinity correspond to the asymptotics $y = x^4$ and $y = -x^4$. It is clear that the involution switches these two asymptics, so it doesn’t fixed any of the points at infinity either.

4. Suppose $X$ as a fixed-point free involution, then the quotient map $X \to X/C = Y$ by the group $C$ generated by the involution is a unramified everywhere double cover of Riemann surfaces. But then $\chi(X) = 2\chi(Y)$, so $2 - 2g_X = 2(2 - 2g_Y)$, so $g_X = 2g_Y - 1$ is odd, a contradiction.

**Problem 4**

Is there a nonconstant holomorphic map from the Fermat quartic, $x^4 + y^4 = 1$, to the octagon curve, $y^2 = x(x^4 - 1)$?

**Solution**

No, there is not. Observe that the Fermat quartic is smooth, and therefore is a canonically embedded curve of genus 3. The octagon curve is a hyperelliptic curve of genus 2. In fact, we’ll show that there is never a map from a non-hyperelliptic curve $X$ of genus 3 to a curve $Y$ of genus 2.

If there were a map from $X$ to $Y$ then the Riemann-Hurwitz formula would give $4 = 2d + \sum (n_p - 1)$, whence $d = 2$ and there are no points of ramification. Therefore, Galois conjugation, $\sigma$, would be a fixed-point free involution of $X$. This is a contradiction by the previous problem.

**Problem 5**

Prove or disprove: if $X$ has genus 3 and $X$ has a canonical divisor of the form $K = 4Q$ for some $Q \in X$, then $X$ is hyperelliptic.

**Solution**

The statement is false. An example is given by the Fermat curve, $X$, of Problem 4, with $Q = P$. If we use the fact that the embedding of the Fermat curve is the canonical embedding, then we know that the linear function $x - z$ corresponds to a one-form on $X$ which vanishes to order 4 at $P$, in other words to a holomorphic
section of $K - 4P$. Since $\deg(K - 4P) = 0$ and it has a holomorphic section, it is trivial, so we see that $K = 4P$.

We can also use Riemann Roch instead of using the fact that the embedding in Problem 4 is canonical. To do it this way, we observe that the functions $1/(x - 1)$ and $y/(x - 1)$ both vanish only at $P$, to orders 4 and 3 respectively. Together with the constant function 1, these give three independent elements of $H^0(4P)$. By Riemann Roch,

$$h^0(K - 4P) = 1 - g + h^0(4P) \geq 1,$$

and from there we see that $K = 4P$.

**Problem 6**

Let $S \subset \mathcal{M}(X)$ be a finite-dimensional subspace of dimension 1 or more, and define a divisor $D = \sum a_P \cdot P$ by

$$a_P = -\min_{f \in S} \text{ord}_P(f).$$

1. Prove that $a_P$ is finite and $a_P = 0$ outside a finite set, so $D$ is in fact a divisor.

2. Prove that for all nonzero $f \in S$, the map $f/g : X \to \mathbb{P}^1$ has degree equal to $\deg D$ for ‘most’ $g \in S$.

**Solution**

1. Choose a basis $\{f_i\}_i$ of $S$. Then because $\text{ord}_P(f + g) \geq \min \{\text{ord}_P(f), \text{ord}_P(g)\}$, we see that $a_P = -\min_i \text{ord}_P(f_i)$. Thus $a_P$ is finite and is zero outside a finite set.

2. Given any point $P$, we can choose a basis $\{f_i\}_i$ of $S$ with the property that $\text{ord}_P(f_i) < \text{ord}_P(f_1)$ for each $i > 1$ (to see this if there are two elements in the basis with the same minimal order at $P$, we can replace one of them by a linear combination that has larger order at $P$). This shows that if $g$ is outside a proper subspace in $S$, $\text{ord}_P(g)$ is equal to the minimal order at $P$ among all elements in $S$.

In particular, for most $g \in S$ we have the above property for all $P$ in the support of $D$. We now compute the degree of $f/g$ for such $g$. A zero of $f/g$ must be either a zero of $f$ or a pole of $g$. At all points $P$ in the support of $D$, we have $\text{ord}_P(f/g) \geq 0$. If $P$ is a zero of $f/g$ not in the support of $D$, then $P$ is not a zero or pole of $g$ and must be a zero of $f$ outside $D$. Thus the number of zeros of $f/g$ is

$$\sum_{f(P) = 0, P \notin D} \text{ord}(f) + \sum_{P \in D} \text{ord}_P(f) - \text{ord}_P(g) = \deg \text{div}(f) + \deg D = \deg D,$$

where the first equality comes from the fact that any pole of $f$ must occur in $D$. 