Problem 1

Let \( P(x) \) be a polynomial of degree \( d > 1 \) with \( P(x) > 0 \) for all \( x \geq 0 \). For what values of \( \alpha \in \mathbb{R} \) does the integral

\[
I(\alpha) = \int_{0}^{\infty} \frac{x^{\alpha}}{P(x)} \, dx
\]

converge? Give a formula for \( I(\alpha) \) in terms of residues. Compute \( I(\alpha) \) for \( P(x) = 1 + x^4 \).

Solution

Let \( d \) be the degree of \( P \). The integral converges when the integrals around a neighborhood around 0 and \( \infty \) both converge, and this happens precisely when \( \alpha > -1 \) and \( \alpha < d - 1 \), because \( P(x) \) is bounded away from 0 near \( x = 0 \) and is \( O(x^d) \) when \( x \to \infty \). We now assume that this is the case.

To write \( I(\alpha) \) in terms of residues, we compute the contour integral of \( f(z) = \frac{z^\alpha}{P(z)} \) around a key-hole shaped contour (as shown below), where we take the branch of \( \log \) defined on the complement of the positive \( x \)-axis with arguments taking values in \((0, 2\pi)\) to define \( z^\alpha \).

The integral over the large circle is bounded by \( O(R^\alpha) R^{-d} R \), which goes to 0 as \( R \to \infty \) by our condition on \( \alpha \). The integral over the small circle is bounded by \( O(r^\alpha) r \), which goes to 0 as \( r \to 0 \) by our condition on \( \alpha \). The integral on the two straight edges will contribute

\[
(1 - e^{2\pi i \alpha}) \int_{0}^{\infty} \frac{x^\alpha}{P(x)} \, dx
\]

up to an error that goes to 0 as \( r \to 0 \) and \( R \to \infty \), because the integral converges. The contour integral is given by the sum of \((2\pi i \times)\) the residues of \( f(z) \) at the poles enclosed by the contour. In the limit this sum of residues stabilizes and involve only terms at the simple zeroes of \( P(z) \) (which in particular does not involve \( z = 0 \)). Thus if \( \alpha \notin \mathbb{Z} \) this gives the formula

\[
I(\alpha) = \frac{2\pi i}{1 - e^{2\pi i \alpha}} \sum_{P(z_i) = 0} \text{Res}_{z_i} \left( \frac{z^\alpha}{P(z)} \right)
\]
Figure 1:
When $\alpha \in \mathbb{Z}$, one can interpret this as the limit of the above formula at the non-integral values of $\alpha$.

For $P(x) = x^4 + 1$, using the above formula we get

$$I(\alpha) = \frac{\pi}{4 \sin((3 - \alpha)/4)}$$

**Problem 2**

Let $X = \mathbb{C} - \{0, 1\}$. Give a pair of closed differential forms on $X$ that furnishes a basis for the de Rham cohomology group $H^1((X, \mathbb{C}) \cong \mathbb{C}^2$.

**Solution**

The forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ are a basis of $H^1$.

**Problem 3**

Let $f(z) = z(e^z - 1)$. Prove there exists an analytic function $h(z)$ defined near $z = 0$ such that $f(z) = h(z)^2$. Find the first 3 terms in the power series expansion $h(z) = \sum a_n z^n$. Does $h(z)$ extend to an entire function on $\mathbb{C}$?

**Solution**

$f$ vanishes to order 2 at 0, so there is a holomorphic function $g$ defined on all of $\mathbb{C}$ such that $f(z) = z^2 g(z)$, and $g(0) \neq 0$. Since $g$ is continuous it is nonzero on some neighborhood $U$ of the origin, shrinking $U$ we may choose a branch of $\log(g(z))$.

Define $h$ on $U$ by

$$h(z) = ze^{\frac{1}{2} \log(g(z))};$$

then $f(z) = h(z)^2$.

The power series expansion of $h$ is

$$h(z) = z + \frac{z^2}{2} + \frac{5z^2}{96}$$

$h$ does not extend to an entire function on $\mathbb{C}$ because such an extension would be a global square root of $f$, which can not exist because $f$ has a simple pole at $2\pi i$.

**Problem 4**

Let $f_t(z)$ be a family of entire functions depending analytically on $t \in \Delta$. Suppose $f_t(z)$ is nonvanishing on $S^1$ for all $t$. Prove that for each $k \geq 0$,

$$N_k(t) = \sum_{|z|<1, f_t(z)=0} z^k$$

is an analytic function of $t$. (The zeroes of $f_t(z)$ are taken with multiplicity).
Solution

By the residue theorem, for each \( t \in \Delta \) we have

\[
N_k(t) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f'_t(z) z^k}{f_t(z)} \, dz
\]

This integral representation makes it clear that \( N_k(t) \) is analytic in \( t \) (for example, by Morera’s theorem).

Problem 5

Let \( B \subset \mathbb{C} \) be the union of circles of radius 1 centered at \( \pm 1 \).

1. Describe the universal cover of \( B \) as a topological space.

2. Draw a connected covering space \( B' \) of \( B \) with deck group \( S_3 \).

3. Draw a picture of \( B'' = B'/A_3 \). What is the deck group of \( B''/B \).

Solution

(see pictures)

1. The universal cover of \( B \) is an infinite 4-valent tree

2. The space \( B' \) will depend on the presentation of \( S_3 \).

3. Since \( A_3 \) is normal in \( S_3 \), the deck group of \( B''/B \) is \( S_3/A_3 \cong \mathbb{Z}/2\mathbb{Z} \).

Problem 6

Let \( X \) be a closed smooth surface of genus two. Let us say a surface \( T \) is large if it arises as a (connected) regular covering space of \( S \) with an infinite deck group. Describe as many different large surfaces as you can (preferably five), and prove that no two surfaces on your list are diffeomorphic to one another.

Solution

We have continuous map from \( X \) to the figure 8, obtained by squishing the handles to the core. Thus one can pullback any connected infinite regular cover of the figure 8 to a large surface \( T \). Using 4 coverings of the figure 8, we construct 4 large surface this way. The universal cover \( \mathbb{H} \) of \( X \) is the fifth large surface. Only \( \mathbb{H} \) is simply connected among these. To distinguish the remaining 4 surfaces from each other, we use the invariants \( g \) and \( E \) of \( T \), as described below.

Let \( T \) be a large surface. For each compact (connected) Riemann surface with boundary \( K \) embedded in \( T \), let \( g_K \) be its genus (the genus \( g \) of a surface with boundary is by definition \( \chi = 2 - 2g - b \) where \( b \) is the number of boundary components
Figure 2:
and $\chi$ is the Euler characteristic). Then the quantity $g := \operatorname{sup}g_K$ is an invariant of $T$.

Let $E$ denote the cardinality of the set $\lim_{\leftarrow K} \pi_0(T \setminus K)$, where the inverse limit runs through all compact subsets of $T$, ordered by inclusion. Then $E$ is another topological invariant of $T$, the number of "topological ends". The following picture shows the 5 large surfaces and how to distinguish them using the above invariants:
Figure 3: