Riemann Surfaces Homework 1

Math 213b — Harvard University — Spring 2001

Due 28 February 2001

1. Let $A \subset X$ be a finite subset of a compact Riemann surface. Suppose $f : (X - A) \to Y$ is a 1-1 holomorphic map to another compact Riemann surface. Show that f extends to an isomorphism $F : X \to Y$.

2. Let X be a compact Riemann surface of genus g and let $A \subset X$ be a finite set with $|A| = n$ even.

   (a) Show there exists a compact Riemann surface Y and a holomorphic map $f : Y \to X$ of degree 2, branched exactly over A.

   (b) What is the genus of Y (in terms of n and g)?

   (c) Draw a (topological) picture of the map $f : Y \to X$ in the case $(g, n) = (1, 2)$.

   (d) In terms of (g, n), how many possibilities are there for $f : Y \to X$, up to isomorphism over X? (Two maps $f_i : Y_i \to X$, $i = 1, 2$ are isomorphic *over* X if there is an isomorphism $g : Y_1 \to Y_2$ such that $f_2 \circ g = f_1$.)

3. Prove that for any sheaf $\mathcal{F}$ over a space X and any open set $U \subset X$, the group $\mathcal{F}(U)$ is naturally isomorphic to the group of continuous sections of $|\mathcal{F}|$ over U.

4. Find a Belyi map $f : X \to \widehat{\mathbb{C}}$ where $X = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}i$ is the square torus. That is, find a holomorphic map to the Riemann sphere branched over just $\{0, 1, \infty\}$.

5. Let $X \subset \mathbb{C}^2$ be a Riemann surface defined the equation $f(x, y) = 0$, where f is a degree d polynomial (meaning the highest order monomials have the form $x^i y^j$, $i + j = d$). Let $\pi : X \to \mathbb{C}$ be projection to the x-coordinate. Show that for 'typical' f, the map π is d-to-1. How many branch points do you expect π to have? (Use resultants.) Derive a formula for the genus of a smooth curve $\overline{X}$ of degree d in $\mathbb{P}^2$. (Use Riemann-Hurwitz.)

6. Show that for any pair of compact Riemann surfaces X_i, $i = 1, 2$, there exists a third compact Riemann surface Y admitting a pair of nonconstant holomorphic maps $f_i : Y \to X_i$. (Hint: think of this as a problem in field theory.)

7. Show for each prime p, there is a unique valuation $v_p : \mathbb{Q}^* \to \mathbb{Z}$ such that $v_p(p) = 1$ and $v_p(q) = 0$ for all primes $q \neq p$. (Thus $v_p(x)$ measures the order to which x 'vanishes at p'.)

   Show that every (discrete) valuation on $\mathbb{Q}$ has this form.

8. Let $\pi : X \to Y$ be a proper map of degree 3 between compact Riemann surfaces. Show that if the field extension $K(Y)/K(X)$ is Galois, then $\text{mult}(\pi, x) \neq 2$ for every $x \in X$. Does the converse hold?