Complex Analysis Homework 9

Math 213 — Harvard University

Due 21 November 2000

1. Let $\langle F_p(z) \rangle$ be a sequence of analytic functions with $F_p(1) = 0$, with no other zeros, and with $F_p(z) \to 1$ uniformly on compact subsets of $\mathbb{C} - \{1\}$. Let $0 \neq a_n \to \infty$ in $\mathbb{C}$. Show there exist p_n such that $G(z) = \prod F_{p_n}(z/a_n)$ converges absolutely on $\mathbb{C}$, giving an analytic function with zeros at $\langle a_n \rangle$ and nowhere else.

2. Prove:

$$\frac{\coth(\pi)}{1^7} + \frac{\coth(2\pi)}{2^7} + \frac{\coth(3\pi)}{3^7} + \ldots = \frac{19\pi^7}{56700}.$$

   (Hint: you may use the fact that the residue of $\cot(\pi z)\coth(\pi z)/z^7$ at $z = 0$ is $-19\pi^6/14175$.)

3. Formulate and prove an infinite product formula for $\cos(\sqrt{z})$. (Here $\cos(\sqrt{z}) = 1 - z/2! + z^2/4! - z^3/6! \cdots$).

4. Find the orders of the entire functions $\cos(\sqrt{z})$ and $\exp(\sin(z))$.

5. Give an example of a function of order 1 of the form $f(z) = \prod_1^\infty (1 - z/a_n)$. (Make sure the product converges.)

6. Let $p(n)$ be the *partition function*; that is, the number of ways to write n as the sum of an increasing sequence of positive integers. (For example $p(5) = 7$ because $5 = 1+1+1+1+1 = 1+1+1+2 = 1+1+3 = 1+4 = 1+2+2 = 2+3$.) Show that

$$1 + \sum_{n=1}^{\infty} p(n) z^n = \prod_{n=1}^{\infty} \frac{1}{1 - z^n}$$

   for all complex z with $|z| < 1$.

7. Let $f(z)$ be an entire function of finite order with simple zeros at the points $z = n + im$, $(n, m) \in \mathbb{Z}^2$. Show there are polynomials P and Q such that $f(z+1) = e^{P(z)} f(z)$ and $f(z+i) = e^{Q(z)} f(z)$. Prove that at least one of P and Q is nonzero.