Complex Analysis Homework 8

Math 213 — Harvard University

Due 14 November 2000

S denotes the space of univalent maps $f : \Delta \to \mathbb{C}$ with $f(z) = z + \sum_2^\infty a_n z^n$.
Σ denotes the space of univalent maps $f : \mathbb{C} - \overline{\Delta} \to \mathbb{C}$ with $f(z) = z + \sum_1^\infty b_n z^{-n}$.

1. Show that if $f, g \in S$ and $f(\Delta)$ contains $g(\Delta)$ then $f = g$.

2. Suppose $f \in S$ satisfies $\sup_\Delta |f(z)| \le M$. Show that $|a_2| \le 2\sqrt{1 - 1/M}$.

   (Hint: $g(z) = f(z^{-2})^{-1/2} \in \Sigma$ omits the ball $B(0, M^{-1/2})$ which has area π/M.)

3. Show that if $f \in \Sigma$ and w is not in the image of f, then $|w| \le 2$.

   (Hint: apply the bound $|b_1| \le 1$ to the function $\sqrt{f(z^2) - w}$.)

4. Prove the *Uniformization Theorem* for bounded domains: show, for any bounded domain $\Omega \subset \mathbb{C}$ (possibly with many holes), there is an analytic *covering map* $f : \Delta \to \Omega$.

   (Idea of the proof: fix $p \in \Omega$. Consider the family $\mathcal{F}$ of all pairs (U, f) where U is a domain in Δ, $f : (U, 0) \to (\Omega, p)$ is an analytic covering map and $f'(0) > 0$. Consider a sequence (U_n, f_n) minimizing $f_n'(0)$ and mimic the proof of the Riemann mapping theorem, to show $(U_n, 0) \to (\Delta, 0)$ in the space $\mathcal{D}$ of Problem 1 on the previous homework.)

5. Prove

$$\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_1^\infty \left(\frac{1}{z - 2n} + \frac{1}{z + 2n} - \frac{1}{z - 2n + 1} - \frac{1}{z + 2n - 1} \right).$$

6. Prove there is no *bounded* analytic function $f : \Delta \to \mathbb{C}$ with zeros at the points $z_n = 1 - 1/n$, $n = 2, 3, 4, \ldots$, and nowhere else. (Hint: consider $f(0)/B_n(0)$, where $B_n(z)$ is a Blaschke product with zeros at $z_2, \ldots, z_n$.)