Complex Analysis Homework 7

Math 213 — Harvard University

Due 7 November 2000

1. Let $\mathcal{R}$ be the space of injective analytic maps $f : \Delta \to \mathbb{C}$ with $f'(0) > 0$, equipped with the topology of uniform convergence on compact sets.

   Let $\mathcal{D}$ be the set of pairs (U, p) where $U \subset \mathbb{C}$ is a simply-connected domain, $p \in U$ and $U \neq \mathbb{C}$. To make $\mathcal{D}$ into a topological space, we say $(U_n, p_n) \to (U, p)$ in D if:

   (i) $p_n \to p$;
   (ii) for every compact set $K \subset U$ we have $K \subset U_n$ for all n sufficiently large; and
   (iii) if V is a domain containing p and $V \subset U_n$ for infinitely many n, then $V \subset U$.

   Define $F : \mathcal{R} \to \mathcal{D}$ by $F(f) = (f(\Delta), f(0))$. Show that F is a homeomorphism.

2. Show that the set of Riemann maps whose images are polygons is dense in $\mathcal{R}$.

3. Let $S = \{x + iy \ : \ (x, y) \in [-1, 1] \times [-1, 1]\}$, and let $f : (\Delta, 0) \to (S, 0)$ be a conformal map. Compute $|f'(0)|$.

4. Show that the region $U = \mathbb{C} - (\infty, -1/4]$ can be thought of as a polygon with two vertices, and use the Schwarz-Christoffel formula to show the Riemann map $f : \Delta \to U$ with $f(0) = 0$ and $f'(0) > 0$ is given by $f(z) = z/(1 - z)^2$.

5. Let $\omega = \omega(z)\, dz$ be a meromorphic 1-form on the Riemann surface. Prove that $\omega = df$ for some rational function $f(z)$ if and only if $\operatorname{Res}(\omega, p) = 0$ for all $p \in \widehat{\mathbb{C}}$.

6. Let $f : U \to V$ be a holomorphic map between domains in $\widehat{\mathbb{C}}$. The *Schwarzian derivative* Sf is the quadratic differential defined by $Sf = Sf(z)\, dz^2$ where

$$Sf(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

   Prove that $Sf = 0$ if and only if f is a Möbius transformation, i.e. iff f is the restriction of an element of $\operatorname{Aut}(\widehat{\mathbb{C}})$.

7. Prove that $S(f \circ g) = Sg + g^*(Sf)$. (Here $g^*(Sf) = Sf(g(z))\, g'(z)^2\, dz^2$.)