Complex Analysis Homework 6

Math 213 — Harvard University

Due 31 October 2000

1. For $r > 0$ let $Q(r) = \{x + iy \ : \ (x, y) \in [0, r] \times [0, 1]\}$ be an $r \times 1$ rectangle in $\mathbb{C}$. Let $f : Q(r) \to Q(s)$ be a homeomorphism sending vertices to vertices and analytic on the interior of $Q(r)$. Prove that $r = s$ or $r = 1/s$.

2. Prove that a conformal map between Jordan domains extends to a homeomorphism between their closures.

3. Define $u(z)$ on S^1 by $u(z) = 1$ if $z^2 \in \mathbb{H}$ and $u(z) = 0$ otherwise. Find a formula for the extension of u to a harmonic function on the disk, and draw the locus where $u(z) = 1/2$.

4. Compute the hyperbolic metric ρ on the annulus $A(R) = \{z \ : \ 1 < |z| < R\}$. That is, find the unique metric such that $f^*(\rho) = |dz|/\operatorname{Im} z$ for any analytic covering map $f : \mathbb{H} \to A(R)$.

5. For $a > b > 0$ consider the ellipse $E \subset \mathbb{C}$ with major axis $[-a, a]$ and minor axis $[-ib, ib]$. Let $I \subset [-a, a]$ be the segment joining the foci of E. Finally let B be the annular region between E and I.

   Find an explicit conformal map $f : A(R) \to B$ for some $R > 1$. (Hint: think about $z + 1/z$.)

6. Let $B \subset \mathbb{C}$ be an annulus bounded by a pair of Jordan curves C_1 and C_2. Prove that there exists an arc $\alpha \subset B$ joining C_1 to C_2, and a loop $\beta \subset B$ separating C_1 from C_2, such that their lengths satisfy

$$L(\alpha)L(\beta) \leq \operatorname{area}(B).$$

   (Hint: let $f : A(R) \to B$ be a conformal map from a standard annulus to B, and consider $\int_{A(R)} |f'(z)| \cdot |z|^{-1} \, |dz|^2$.)