Complex Analysis Homework 5
Math 213 — Harvard University
Due 24 October 2000

1. Find formulas for the following analytic isomorphisms:
   (a) $f : \mathbb{H} \to \Delta$;
   (b) $f : \mathbb{H} \to \{z \in \mathbb{C} \,:\, 0 < \operatorname{Re} z < 1\}$;
   (c) $f : \{z \in \mathbb{H} \,:\, -\pi < \operatorname{Re} z < \pi\} \to \mathbb{H}$;
   (d) $f : (\mathbb{C} - \overline{\Delta}) \to \mathbb{C} - [-2, 2]$.

2. Let $f_n : U \to \mathbb{C}$ be a sequence of analytic functions converging uniformly on compact sets to a nonconstant function $f : U \to \mathbb{C}$. Assume the mappings $f_n(z)$ are at most d-to-1. Show the same is true of $f(z)$.

3. Give an example of a sequence of degree 3 rational maps $f_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $f_n(S^1) \subset S^1$, and $f_n(z) \to z$ uniformly on S^1. Compute $\int_{S^1} f_n'(z)/f_n(z)\, dz$ for your example.

4. Prove that for any $p \geq 1$, the analytic functions

$$\mathcal{F} = \{f : \Delta \to \mathbb{C} \,:\, \int |f'(z)|^p |dz|^2 \leq 1\}$$

   form a normal family.

5. Let $\rho = 2|dz|/(1 + |z|^2)$ be the *spherical metric*. Show that a transformation $f(z) = (az + b)/(cz + d) \in \operatorname{Aut} \widehat{\mathbb{C}}$ preserves the metric ρ iff f lies in the image of $SU_2(\mathbb{C})$ in $PSL_2(\mathbb{C})$; that is, iff the coefficients of f can scaled so that the matrix $F = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ preserves the norm $\|(z_1, z_2)\|^2 = |z_1|^2 + |z_2|^2$ on $\mathbb{C}^2$.

6. Compute the hyperbolic area of an ideal triangle. That is, compute the area, in the metric $\rho = |dz|/\operatorname{Im} z$, of the region $\Omega \subset \mathbb{H}$ given by

$$\Omega = \{z \in \mathbb{H} \,:\, |z| > 1 \text{ and } |\operatorname{Re} z| < 1\}.$$