Complex Analysis Homework 4
Math 213 — Harvard University
Due 17 October 2000

1. Let $f(z)$ be analytic and bounded on $\Delta - E$, where E is a compact set of 1-dimensional measure zero. (This means for any $\epsilon > 0$ there is a finite covering of E by balls $B(z_i, r_i)$ with $\sum r_i < \epsilon$.) Prove that $f(z)$ extends to an analytic function on the whole disk Δ.

2. Let $p(z)$ be a polynomial of degree $d > 1$. Show that the zeros of $p'(z)$ belong to the convex hull of the zeros of $p(z)$. (The convex hull of a set E is the intersection of all the convex sets containing E.)

   Hint: consider $p'(z)/p(z)$.

3. Prove that any proper holomorphic map $f : \Delta \to \Delta$ of degree two can be written in the form $f(z) = A(S(B(z)))$, with $A, B \in \operatorname{Aut}\Delta$ and $S(z) = z^2$. (Here $\Delta = \{z \;:\; |z| < 1\}$.)

4. Let $\Gamma \subset \operatorname{Aut}(\widehat{\mathbb{C}})$ be a finite group. A rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a *quotient map* for Γ if $f(x) = f(y)$ iff $\gamma(x) = y$ for some $\gamma \in \Gamma$.

   Find quotient maps f_1 and f_2 for the groups of order two generated by $\gamma_1(z) = -z$ and $\gamma_2(z) = 1/z$.

   Draw the images of the circles $|z| = r$, $r > 0$ under $f_2(z)$.

5. Let $p(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3$ be a cubic polynomial. Assume $a_0 \neq 0$ and $p'(z)$ has distinct zeros.

   (a) Show that we can write $A(p(B(z))) = C(z) = z^3 - 3z$ for suitable $A, B \in \operatorname{Aut}\mathbb{C}$. (Hint: adjust p so its critical points are at ± 1.)

   (b) Show how to solve the equation $z^3 - 3z = w$ by radicals, using the fact that if $z = a + a^{-1}$, then $z^3 - 3z = a^3 + a^{-3}$.

   (c) Combining the previous calculations, solve the equation $z^3 + az + b = 0$ by radicals.

6. Suppose a pair of analytic functions on a region U satisfy $|f(z)| = |g(z)|$. Prove that $f(z) = ag(z)$ for some constant a with $|a| = 1$.

7. Let $M(r) = \sup_{|z|=r} |f(z)|$ where $f : \mathbb{C} \to \mathbb{C}$ is an analytic function, not identically equal to zero. Suppose $M(r^2)^2 = M(r)M(r^3)$ for some $r > 0$. Prove that $f(z) = az^n$ for some $a \neq 0$ and integer $n \geq 0$.

8. Prove that there is no conformal metric $\rho = \rho(z)|dz|$ on $\widehat{\mathbb{C}}$ that is invariant under the full automorphism group, $\operatorname{Aut}(\widehat{\mathbb{C}})$.