Complex Analysis Homework 3

Math 213 — Harvard University

Due 10 October 2000

1. Prove the identity

$$\int_0^{2\pi} \cos^{2n}\theta \, d\theta = 2\pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

   by applying the residue theorem to $\int_{S^1} (z + 1/z)^{2n} \, dz/z$.

2. Given $w \in \mathbb{C}$, find an explicit sequence $z_n \to 0$ such that $e^{1/z_n} \to w$.

3. Show that the polynomial $z^5 + 15z + 1$ has all its roots in the region $|z| < 2$, but only one root lies in the region $|z| < 3/2$.

4. Consider the equation

$$\int_0^{\infty} \frac{x^{\alpha}\, dx}{(1+x^2)^2} \quad = \quad \frac{\pi(1-\alpha)}{4\cos\frac{1}{2}\pi\alpha}.$$

   Find the set of real α such that the integral above is absolutely convergent, and prove the equation above holds for all such α.

5. Let $f(z)$ be analytic in a region Ω, and suppose $\int_{\Omega} |f(z)|\,|dz|^2 < \infty$. Prove that there is a constant M such that for all $z \in \Omega$ we have

$$|f(z)| \leq \frac{M}{d(z, \partial\Omega)^2},$$

   where $d(z, \partial\Omega) = \inf\{|z - w| \ : \ w \in \partial\Omega\}$.

6. Compute the first four nonzero terms in the power series for $\tan(z)$ at $z = 0$ by formally inverting the power series for $\tan^{-1}(z) = \int dz/(1+z^2)$.

7. Compute

$$\int_{|z|=1/2} \frac{dz}{z \sin^2(z)}.$$