Complex Analysis Homework 13

Math 213 — Harvard University

Due 19 December 2000

1. Prove that $g_3(i) = 0$ and $g_2(\omega) = 0$, where $\omega = \exp(2\pi i/3)$. (Recall $g_k(\tau)$ is proportional to $\sum_{\Lambda^*} \lambda^{-2k}$, with $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$.)

2. Prove that $J(z) = g_2(z)^3/(g_2(z)^3 - 27g_3(z)^2)$. (Recall
$$J(z) = \frac{4}{27}\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1-\lambda)^2},$$
and $\lambda(z)$ is the cross-ratio of the roots of the cubic equation $4x^3 - g_2(z)x - g_3(z) = 0$, together with ∞.)

3. Prove that $\lambda(i/2) = 12\sqrt{2} - 16$.

   (Hint: Letting $X_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$, $\lambda(\tau)$ is the cross-ratio of the critical values (suitably ordered) of any degree two map $f_\tau : X_\tau \to \widehat{\mathbb{C}}$. For the square torus, $\tau = i$, choose f_i so its critical values are the roots of $z^4 + 1 = 0$. Then one can choose $f_{i/2}(z) = (f_i(z) + f_i(z)^{-1})/2$, and find that its critical values are $\{-1, -\sqrt{1/2}, \sqrt{1/2}, 1\}$.)

4. For any $\lambda \neq 0, 1, \infty$ let $S_\lambda = \widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}$. Let $\operatorname{Aut}(S_\lambda)$ be the group of holomorphic bijections $f : S_\lambda \to S_\lambda$.

   (a) Prove that every $f \in \operatorname{Aut}(S_\lambda)$ is a Möbius transformation.

   (b) Prove that for all λ, $\operatorname{Aut}(S_\lambda)$ contains a subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. How does this group permute the points $\{0, 1, \infty, \lambda\}$?

   (c) Find all values of λ such that $|\operatorname{Aut}(S_\lambda)| > 4$ and identify the group in each case.

5. Let $\rho(z)\,|dz|$ be the hyperbolic metric on $X = \widehat{\mathbb{C}} - \{0, 1, \infty\}$; that is, the unique metric such that $\lambda : \mathbb{H} \to X$ is a local isometry. Show there are constants $A, B > 0$ such that for small values of z,
$$\frac{A}{|z \log |z||} \leq \rho(z) \leq \frac{B}{|z \log |z||}.$$
(Hint. Recall $|dz|/|z \log |z||$ is the hyperbolic metric on $\Delta^* - \Delta - \{0\}$. Using the fact that $\lambda(z + 2) = \lambda(z)$, write $\lambda(z) = f(\exp(\pi i z))$ where $f : \Delta^* \to X$ is a covering map. Observe that $f(0) = 0$ and $f'(0) \neq 0$ to complete the proof.)

6. Let L be length in the hyperbolic metric of the closed geodesic γ on X that makes a figure 8 around 0 and 1. Show that $L = \log(17 + 12\sqrt{2})$.