Complex Analysis Homework 12

Math 213 — Harvard University

Due 12 December 2000

1. Let $\Lambda \subset \mathbb{R}^n$ be a lattice, i.e. a discrete subgroup isomorphic to $\mathbb{R}^n$. Choose a sequence of vectors $a_1, a_2, \ldots, a_n \in \Lambda$ such that a_1 is a shortest nonzero vector, and (for $i > 1$) a_i is a shortest vector linearly independent from $(a_1, a_2, \ldots, a_{i-1})$.

   Is it always the case then that $\Lambda = \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_n$?

2. Let $\Lambda \subset \mathbb{C}$ be a lattice, let $X = \mathbb{C}/\Lambda$ and let $\mathrm{End}(\Lambda) = \{\alpha \in \mathbb{C} \ : \ \alpha\Lambda \subset \Lambda\}$. Show that for each $\alpha \in \mathrm{End}(\Lambda)$, the formula $[f(z)] = [\alpha z]$ defines an analytic covering map $f : X \to X$ of degree $|\alpha|^2$.

   For what values of $\alpha \in \mathbb{C}$ does there exist a lattice with $\alpha \in \mathrm{End}(\Lambda)$?

   Conclude that $\mathrm{End}(\mathbb{Z} \oplus \mathbb{Z}\tau) = \mathbb{Z}$ for almost all values of τ.

3. State and prove a 'double angle' formula for the Weierstrass $\wp$-function. That is, find a rational function $f(z)$ (that may depend on (g_2, g_3)) such that $\wp(2z) = f(\wp(z))$.

4. Let $X = \mathbb{C}/\Lambda$ be a complex torus, and define a map $F : X \to X$ by $[F(z)] = [2z]$. Show that F has a dense orbit on X, i.e. that there exists a $p \in X$ such that $\overline{\{F^n(p) \ : \ n > 0\}} = X$, where $F^n(p) = F(F(\cdots F(p)))$.

   Then prove the rational function $f(z)$ of the double-angle formula has a dense orbit on $\widehat{\mathbb{C}}$.

5. Prove that $\wp'(z) = -\sigma(2z)/\sigma(z)^4$.

6. Let $T \subset \mathbb{C}$ be the region bounded by the triangle with vertices $(0, 1, 1 + i)$. Given an explicit formula for a conformal mapping $f : T \to \mathbb{H}$ in terms of the Weierstass $\wp$-function for a suitable lattice.