Complex Analysis Homework 10/11
Math 213 — Harvard University
Due 5 December 2000

Notation: $\mathcal{O}(\mathbb{C})$ denotes the ring of analytic functions on $\mathbb{C}$.

1. Find all entire functions $f(z)$ satisfying $f(z+1) = 2f(z)$. Show there are countably many such functions of finite order with no zeros and with $f(0) = 1$.

2. What is the order of the entire function $1/\Gamma(z)$?

3. For a fixed value of $a > 0$, define for $\operatorname{Re} s > 0$ the function
$$F(s) = \int_0^1 x^s (1-x)^a \frac{dx}{x(1-x)}.$$
Prove $F(s) = \Gamma(a)\Gamma(s)/\Gamma(a+s)$. (Hint: show $G(s) = F(s)\Gamma(a+s)$ satisfies the functional equation $G(s+1) = sG(s)$.)

4. Using the previous result, evaluate $\int_0^1 (1-x^a)^b\, dx$ for $a, b > 0$.

5. Finally evaluate $|f_n'(0)|$, where $f_n : (\Delta, 0) \to (P_n, 0)$ is the Riemann map from the unit disk to the regular polygon P_n with vertices at the nth roots of unity.

6. State and prove a necessary and sufficient condition for a meromorphic 1-form $\omega = \omega(z)\, dz$ on $\mathbb{C}$ to be the logarithmic derivative, $\omega = d \log f = f'(z)/f(z)\, dz$, of a meromorphic function $f(z)$.

7. Prove that for any sequences $a_n, b_n \in \mathbb{C}$ with $a_n \to \infty$, there exists an entire function such that $f(a_n) = b_n$. (Hint: write $f(z) = \sum g_n(z)$ where g_n is a polynomial constructed by induction, such that $g_n(a_i) = 0$ for $i < n$, $g_n(a_n) = b_n - \sum_1^{n-1} g_i(a_n)$, and $|g_n(z)| < 2^{-n}$ when $|z| < |a_n|/2$.)

8. Show that if $f, g \in \mathcal{O}(\mathbb{C})$ have no common zeros, then $(f, g) = (1)$; i.e. show there exist $r, s \in \mathcal{O}(\mathbb{C})$ such that $fr + gs = 1$.

   (First reduce to the case where f and g have simple zeros, using the fact that $(f, g) = (f + ag, g)$; then use Problem 7.)

9. Let $I = (f_1, f_2, \dots) \subset \mathcal{O}(\mathbb{C})$ be the ideal generated by the sequence of functions $f_n(z) = \sin(z/n)/z$. Prove that I is not contained in any proper principal ideal. Conclude that $\mathcal{O}(\mathbb{C})$ is not a PID and that $\mathcal{O}(\mathbb{C})$ contains maximal ideals that are not of the form $(z - a)$.