Complex Analysis Homework 1
Math 213 — Harvard University
Due 26 September 2000

1. Let $f_n \to f$ uniformly on compact subsets of an open connected set $\Omega \subset \mathbb{C}$, where f_n is analytic, and f is not identically equal to zero.

   (a) Show if $f(w) = 0$ then we can write $w = \lim z_n$, where $f_n(z_n) = 0$ for all n sufficiently large.

   (b) Does this result hold if we only assume Ω is open?

2. Let $f(z) = (az+b)/(cz+d)$ be a Möbius transformation. Show the number of rational maps $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

$$g(g(g(g(g(z))))) = f(z)$$

   is 1, 5 or ∞. Explain how to determine which alternative holds for a given f.

3. Find all meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ such that $f(1) = 1$ and

$$f(z+1) = zf(z).$$

4. Let $\sum a_n z^n$ be the Taylor series for $\tan(z)$ at $z = 0$.

   (a) What is the radius of convergence of this power series?

   (b) Give an explicit value of N such that $\tan(1)$ and $\sum_0^N a_n$ agree to 1000 decimal places. Justify your answer.

5. Evaluate:

$$\int_{-\infty}^{\infty} \frac{x^6}{(1+x^4)^2}\, dx.$$

6. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and let $U \subset \mathbb{C}$ be a bounded region. Suppose $|f(z)|$ is constant on ∂U. Show that either f is constant, or f has a zero in U.

7. Compute the Laurent series centered at $z = 0$ such that

$$\sum_{-\infty}^{\infty} a_n z^n = \frac{1}{z(z-1)(z-2)}$$

in the region $1 < |z| < 2$.

8. Show for any polynomial $p(z)$ there is a z with $|z| = 1$ such that $|p(z) - 1/z| \geq 1$.

9. Let $U = \{z \ : \ 0 < |z| < 1 \text{ and } 0 < \arg(z) < \alpha\}$, where $0 < \alpha < 2\pi$ (see below). Find a formula for a conformal homeomorphism $f : U \to \mathbb{H}$, where $\mathbb{H} = \{z \ : \ \operatorname{Im}(z) > 0\}$ is the upper half-plane.

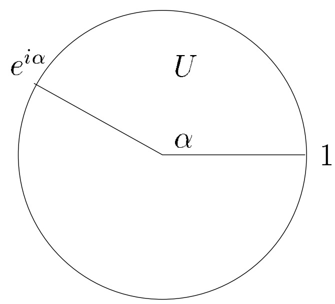

Figure 1. Pie slice.

10. Express

$$f(z) = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^4}$$

in closed form, using trigonometric functions.