SOLUTION SET #9

9.51(3)

a). Let $F\colon I\times I\to I$ be defined by $(t,x)\mapsto(1-t)x$. Let $G\colon \mathbb{R}\times I\to\mathbb{R}$ be defined by $(x,t)\mapsto(1-t)x$. These are homotopies between the identity and the constant map at 0, so both spaces are contractible.

b). Recall that if there is a path between a,b and a path between b,c, then there is a path between $a,\&c$. It therefore suffices to show that all points can be connected to a given point by a path. Since X is contractible, there is a homotopy between 1_X and the constant map at some point x_0. Denote this map F. Out path between a given point x and x_0 is just $F(x,\cdot)\colon I\to X$.

c). Recall that if $f\sim g$, then $f\circ h\sim g\circ h$ for any h with the appropriate range. Given $f\in[X,Y]$, we know that $f=1_Y\circ f$, and since Y is contractible, $1_Y\sim c_{y_0}$ for some y_0. This means $f\sim c_{y_0}\circ f=c_{y_0}$. In other words, every f is homotopic to this constant map, so $[X,Y]$ is one point.

d). It is easy to see that a path is the same thing as a homotopy between constant maps. If X is contractible, then since $f=f\circ 1_X$, and $1_X\sim c_{x_0}$, we have that any map is homotopic to $c_{f(x_0)}$, but all constant maps are homotopic, since this is a path connected space, so we are done.

9.52(3)

If $\pi_1(X,x_0)$ is abelian, then since $g=\alpha*\bar\beta$ is a loop in $\pi_1(X,x_0)$, we have that $gfg^{-1}=f$, and so

$$[\alpha]*[\bar\beta]*[f]*[\beta]*[\bar\alpha]=\hat\alpha\hat{\bar\beta}[f]=[f].$$

This means $\hat{\bar\beta}=\hat\alpha^{-1}$, and the first result follows.

Now notice that given ANY path β between the two points, $\alpha=g*\beta$ is another path. Now just use the fact that $\hat\alpha\hat{\bar\beta}=1$, apply this identity to f and expand both sides.

9.52(4)

Let r be the retraction $X\to A$, and let ι be the inclusion $A\to X$. By the way this is set up, $r\circ\iota=1_A$. This means that $r_*\circ\iota_*=1_{A*}=1_{\pi_1(A)}$. The right hand side is an isomorphism, so we immediately conclude that r_* is a surjection and ι_* is an injection.

9.52(5)

Let $\iota\colon A \to \mathbb{R}^n$ be the inclusion map, and let $\tilde{h}\colon \mathbb{R}^n \to Y$ be the extension of h. Then $h = \tilde{h} \circ \iota$. That means that h_* is just $\tilde{h}_* \circ \iota_*$. Since $\pi_1(\mathbb{R}^n) = 0$, we conclude that h_* must be trivial:

$$\pi_1(A, a_0) \xrightarrow{\iota_*} \pi_1(\mathbb{R}^n, a_0) = 0 \xrightarrow{\tilde{h}_*} \pi_1(Y, y_0).$$

9.53(3)

Let U_k be the set of elements of B such that $|p^{-1}(b)| = k$, and let U_∞ be the set of all elements of B with an infinite preimage under p. By even covering of the map p, if $b \in U_k$, then a neighborhood of b is in U_k, so each of these sets are open. Thus U_k and $\bigcup_{n \neq k} U_n$ are both open and are disjoint. Since B is connected, this means one of these sets is empty, and since $|p^{-1}(b_0)| = k$, we are done.

9.53(5)

Everyone got this right. Just take a very small neighborhood of any point $x \in S^1$, and the preimage under p_n is just the same set translated by $\frac{2\pi k}{n}$.

9.54(4,5)

There are a large number of lifts of each map, so I will just give one for each:

$f(t)$ $\tilde{f}(t) = (0, 2-t)$.
$g(t)$ $\tilde{g}(t) = (t, t+1)$.
$h(t)$ $\tilde{h}(t) = \tilde{f} * \tilde{g}$.
$k(t)$ $\tilde{k}(t) = (t, 2t)$

9.54(6)

We can represent the generator of $\pi_1(S^1, 1)$ by the map $t \mapsto e^{2\pi i t}$. With this generator, using path lifting, we see that the element $k \in \mathbb{Z} \cong \pi_1(S^1, 1)$ is represented by $t \mapsto e^{2\pi i k t}$. The maps g_* and h_* therefore take k to $(t \mapsto e^{2\pi i n k t}) = nk$. So they are multiplication by n, (with the appropriate sign).