1. Closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$

We know that a point x is closure of $\mathbb{R}^\infty$ iff every ball about it intersects $\mathbb{R}^\infty$. This is the same as $\forall \epsilon > 0$, there is a $y \in \mathbb{R}^\infty$ such that $y \in B_{\bar{\rho}}(x, \epsilon)$. Since any element of $\mathbb{R}^\infty$ is eventually 0, this means that eventually, all of the terms of x must be within ϵ of 0. So we conclude that the closure is just the set of all points x such that $x_n \to 0$.

As an aside, $\mathbb{R}^\infty$ can be viewed as the "limit" object of the sequence of spaces $\mathbb{R} \subset \mathbb{R}^2 \subset \dots$, where we include $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ as $x \mapsto (x, 0)$. It is interesting to compute the closure of $\mathbb{R}^\infty$ in the product topology as well...

2. Balls in the uniform topology aren't what you would expect

Many people noticed that part a follows immediately form part b, and part b simply requires that we take a point x' such that $|x'_n - x_n| \to \epsilon$. Pretty much any x' will work (like, say $x'_n = x_n + \frac{n}{n+1}\epsilon$). There is no ball of radius δ about x' contained entirely in U for any δ, so we conclude that U in fact cannot be open.

Part c followed with a small amount of work. First, if y is in the union of the U_δ, then it is in [at least] one of them, say U_δ. This means that $\sup |y_n - x_n| \leq \delta < \epsilon$, so $y \in B(x, \epsilon)$. Conversely, if $y \in B(x, \epsilon)$, then we know that $|y_n - x_n| \leq \delta_1 < \epsilon$ for some δ_1. Pick a δ between δ_1 and ϵ, and y lies in this U_δ.

3. Uniform convergence allows us to do what is clear

Pick $\epsilon > 0$. We know that $f_n \to f$ UNIFORMLY. This means that for $\epsilon/2$, we can find an N such that for all $n > N$, and $x \in X$, $d(f_n(x), f(x)) < \epsilon/2$ (note here that I did not take off points if you wrote the metric on Y as $|\cdot|$, but if I see this again...) In particular, $d(f_n(x_n), f(x_n)) < \epsilon/2$ for all $n > N$. Similarly, since the limit is the uniform limit of continuous functions, it is continuous and we can conclude that since $x_n \to x$, there is some M such that for all $m > M$, $d(f(x_m), f(x)) < \epsilon/2$. Take $N' = \max\{M, N\}$. Then by the triangle inequality, for all $n > N'$, $d(f_n(x_n), f(x)) \leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(x)) < \epsilon$.

4. Example

The first part was just standard calculus. For any fixed x, the rational function $f_n(x)$, viewed as a function of n has no positive singularities, so we can just use calculus to conclude that the limit is 0.

For the second part, notice that $f_n(\frac{1}{n}) = 1$ for all n. For the convergence to be uniform, given an $\epsilon > 0$ there must be an N such that for all $n > N$ and for all $x \in \mathbb{R}$, we must have $|f_n(x)| < \epsilon$. But if we take $\epsilon < 1$, we immediately get a contradiction.

5. Comparing Topologies

If X is connected in the finer topology, then it is obviously connected in the coarser topology. About the converse we can really say nothing. The trivial topology on any space and the discrete topology provide one simple counter example.

6. Stronger Fan Lemma

Define $B_n = A_1 \cup \cdots \cup A_n$. By induction, since B_1 is connected, $B_n = B_{n-1} \cup A_n$, $A_{n-1} \cap A_n \neq \varnothing$, and the fan lemma, we conclude that all of the B_n are connected. Finally, since $\bigcap B_n = A_1 \neq \varnothing$, we can apply the fan lemma to conclude that $B_\infty = \bigcup B_n = \bigcup A_n$ is connected.

7. Totally Disconnected

Every set is open. So given a subset A, if $|A| > 2$, then for any point $x \in A$, $\{x\} \coprod A - \{x\}$ forms a separation of A. So the only connected sets are single points.

Not every totally disconnected set is discrete. $\mathbb{R}_l$ is totally disconnected, as is $\mathbb{Q}$ with its standard topology.

8. Connectedness of Products

A lot of people tried to just take the union of all the sets $X \times \{y\} \cup \{x\} \times Y$. Yes. These sets are connected. And yes, in general there is STRONG version of the fan lemma which guarantees that this set will be connected. But you don't have that version.

The easiest way to do this is to instead pick a point $(a, b) \in (X - A) \times (Y - B)$. We can look at all all sets of the form $S_v = X \times \{v\} \bigcup \{a\} \times Y$. These all have non-empty intersection, so their union is connected since all of them are. Similarly, take $T_u = X \times \{b\} \cup \{u\} \times Y$, and this is a "fan-lemmable" collection of connected sets. If we take the union of the $S_v's$ over $v \in Y - B$, and the union of the $T_u's$ over $u \in X - A$, then we can union these two sets together as well. Since (a, b) is in both, we know this is a connected union, so we are done.

9. Homeomorphisms

First, we can remove 2 points from $[0, 1]$ without disconnecting it. We can only remove 1 point from $[0, 1)$ without disconnecting it, and we cannot remove any points from $(0, 1)$ without disconnecting it. This "number of points" property is a homeomorphism invariant, so we conclude that none of these are homeomorphic.

From part a we can get part b. $(0, 1)$ and $[0, 1]$ are not homeomorphic, but we can pretty clearly imbed each in the other.

Finrally, we can use the same idea to get part c. $\mathbb{R}^n - \{0\}$ is connected, so the image in $\mathbb{R}$ must be connected, but $\mathbb{R}$ minus a point is disconnected. For the interested, this type of procedure CAN be generalized to show that $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $m \neq n$.

10. Fixed Point Theorem

First, if $f(0) = 0$, or $f(1) = 1$, then the result is obvious. Assume to the contrary. Now define a function $g(x) = f(x) - x$. Then $g(0) > 0$, and $g(1) < 0$. Since $[0, 1]$ is connected, we can apply the intermediate value theorem to conclude that there is a point where g is 0. Thus f has a fixed point.

We can relatively simply construct examples of how this fails. i.e. On $(0, 1)$ take $x \mapsto x^2$.