SOLUTION SET 11

9.59.2

Everyone got this right. There need not be a point in the compliment of the image of a loop.

9.59.3

a) was done on a previous problem set. b) can be done similarly: $\mathbb{R}^2 - p \simeq S^1$ is not simply connected, but $\mathbb{R}^n - p \simeq S^{n-1}$ is for $n > 2$. Simple connectivity is a homeomorphism invariant.

9.60.1

$S^1 \times B^2 \simeq S^1$, so $\pi_1(S^1 \times B^2) = \mathbb{Z}$. $\pi_1(S^2) = 0$, so $\pi_1(S^1 \times S^2) = \pi_1(S^1) \times 0 = \mathbb{Z}$.

9.60.2

Embed B^2 in S^2 as the upper hemisphere. It will be sufficient to show that every element in $\mathbb{RP}^2$ has a preimage in B^2. This follows immediately, since any point in the southern hemisphere minus the equator has a corresponding point in the northern hemisphere, and any point on the equator is already double covered. In other words, when we do the identifications, everything will clearly still work out fine.

9.60.4

The space $\mathbb{RP}^1 = S^1$, and the covering map is just the map that sends $\pm x \to x^2$.

9.60.5

For this we use path lifting. Call the point in the diagram on the left a and the one on the right b. If we lift the path AB, we reach one, and if we life BA we reach the other. By the uniqueness of path liftings, if these maps are to be homotopic, then these points should be the same.

11.68.2

a). let $a \in G_1$, $b \in G_2$. Then ab and ba are reduced words, so by the uniqueness condition built into free products, they must represent distinct elements.

b). Clearly, by looking at a reduced word representing x, if x has even length, then x^k is still a reduced word of greater length, so $x^k \neq 1$ for all $k > 0$. If the length of x is odd, and a_1 is the first element in a reduced word expansion of x and a_{2n+1} is the last, then a_1, a_{2n+1} are in the same prodicand group G_1, G_2. So $a_1^{-1} x a_1$ will have strictly shorter length.

c). Clearly, all of the elements identified have finite order. We must only show that there are no others. Assume that x is not of this form. We know, since x has finite order, that the length of x is not even. Since finite orderness of x is unchanged by conjugation, this means that for any conjugate y of x, $len(y)$ is not even. By the previous, x is conjugate to something of smaller length. This in turn implies (induction!) that eventually we can show that some conjugate of x is an element of length 1, and this means that some conjugate of x is in G_1 or G_2 and has finite order. Contradiction.

11.68.3

We can see this immediately by looking at lengths. If $c \in G_1$, then the intersection is clearly trivial, since $G_1 \cap G_2 = (0)$. If $c \in G_2$, then the element cxc^{-1} is a reduced word for all $x \neq 0 \in G_1$. Finally, in the general case, we see by looking at a reduced word representation of c that the length of cxc^{-1} is greater than 1. Since every element of G_2 has length 1, this is a contradiction.

11.69.3

a). The abelization of G is just $Z_m \times Z_n$, and this has order mn.

b). By problem 11.68.2, we know that the only elements of finite order are those in G_1, or G_2 and their conjugates. This means that k is just the max of the orders of elements in G_1, G_2. So $k = \max(m, n)$.

c). Now m, n are uniquely determined. One is k and the other is just $|G/[G,G]|/k$.

11.69.4

Ok. I will not do this in the full generality, though the example i give generalizes obviously. Let $G_1 = \mathbb{Z}/2\mathbb{Z}$, $G_2 = \mathbb{Z}/15\mathbb{Z}$. The direct sum of these groups is isomorphic to $\mathbb{Z}/30\mathbb{Z}$, by the map $(1,1) \mapsto 1$. In the same way, we see that this group is isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$.