SOLUTION SET 10

9.55.1

Let $r\colon B^2 \to A$ be the retract. Then we know, since $A \subset B^2$, that the map $\tilde{f} = f \circ r\colon B^2 \to A$ can be realized as a map $B^2 \to B^2$. This necessarily has a fixed point a, and this fixed point must lie in A. But since r is a retraction, $a = f \circ r(a) = f(a)$, and so f has a fixed point.

9.55.2

Since h is nulhomotopic, it extends to a map $\tilde{h}\colon B^2 \to S^1$. $S^1 \subset B^2$, and so we know that the map $\tilde{h}$ must also have a fixed point, and this fixed point must lie in S^1, since the image is contained in S^1. But $\tilde{h}|_{S^1} = h$, so the first part is shown.

For the second, if h is nulhomotopic, then $H = h \circ I$, where I is the "antipode" map is also nulhomotopic. Apply the previous part.

9.56.1

Pretty much everyone got this right. You need only notice that for g defined as in the hint, $|g(x)| > 1 - (|a_0| + \dots) > 0$, for all $x \in B^2$. From this, the result follows immediately.

9.57.1

View the Earth as S^2, and for a point $p \in S^2$, let $t(p)$ be the temperature and $b(p)$ the barometric pressure. Then the function $p \mapsto (t(p), b(p))$ is a continuous map $S^2 \to \mathbb{R}^2$, and so by the Borsuk-Ulam theorem, there is a point where the value of this function is the same as the value at the points antipode.

9.57.4

a). Assume that there is such a retraction, $r\colon B^{n+1} \to S^n$. Then $r|_{S^n} = Id_{S^n}$, by the properties of a retraction, and the map $R\colon S^n \times I \to S^n$ given by $R(x,t) = r(xt)$ shows that the identity is nulhomotopic.

b). Embed S^n in S^{n+1} as the equator. Then if such a g exists, we have that $g|_{S^n}$ is a continuous, antipode preserving map $S^n \to S^n$. Now we use the same procedure: the map g provides an extension of $g|_{S^n}$ over a copy of B^{n+1}, and this implies that the map $g|_{S^n}$ is nulhomotopic (we just use the same idea as before).

c). Assume that there is no point where $f(x) = f(-x)$. Then the map given by $x \mapsto \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}$ is a continuous, antipode preserving map $S^{n+1} \to S^n$, contradicting b).

d). This is proved in EXACTLY the same way as the bisection theorem in the book.

9.58.2

a) $B^2 \times S^1 \simeq S^1$, so $\pi_1 = \mathbb{Z}$.
b) $T^2 - p \simeq S^1 \vee S^1$ (just view the square with edges identified and then project radially out), so $\pi_1(T^2 - pt) = \mathbb{Z} * \mathbb{Z}$.
c) trivial
d) trivial
e) take some point like $(-1, -1, -1)$ as the base point, and the loop around the positive x axis and the loop around the y axis provide two non-homotopic generators of the fundamental group, so it has the fundamental group of the figure 8.
f) $\mathbb{Z}$, deformation retracts to the circle.
g) $\mathbb{Z}$, deformation retracts to the circle.
h) trivial.
i) $\mathbb{Z}$, crush to the circle.
j) $\mathbb{Z}$, crush to the circle.
k) $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$.
l) trivial.

9.58.4

This was done in class. Crush the bar of the theta to a point, and for the other map, just take the central arcs of the figure 8 and crush them to the center.

9.58.5

First, if a space is contractible, then there is some x_0 such that $1_X \simeq c_{x_0}$. This immediately implies that this has the homotopy type of a point.

Second, if it has the homotopy type of a point, then there are maps $f\colon X \to y$, $g\colon y \to X$, y a point, such that $f \circ g \simeq 1_X$. but $f \circ g$ is constant, so Id_X is homotopic to a constant map.

9.58.9

a). This follows because all of the groups $\pi_1(S^1, x_0)$ are CANONICALLY isomorphic. Just pick the shortest path between any two points and chase the diagram.

b). This follows because $h \simeq k$ implies that $h_* = k_*$, and by the previous, we can, without loss of generality, assume that $h(x_0) = k(x_0)$.

c). $(h \circ k)_* = h_* \circ k_*$, so we get that $(h \circ k)_*(\gamma) = h_*(k_*(\gamma)) = deg(h)deg(k)\gamma$.

d).

Constant	0
Ident	1
Reflection $= z \mapsto z^{-1}$	-1
$z \mapsto z^n$	n

The last two follow from the last part of problem set 9.

e). This is the hardest part. Again, we can reduce to the case that $h(x_0) = k(x_0)$. Now the problem is made much easier if we use a slightly different notion of the fundamental group. Recall that $\pi_1(X)$ is the homotopy classes of paths with $f(0) = f(1)$. These are represented by maps from $I \to X$ with this property. Since the ends of the interval MUST map to the same point, we can really realize any of these as functions from the interval with the two endpoints identified to the space X, where the fat endpoint must always be mapped to some point x_0. In other words, we can realize this as the homotopy classes of maps $S^1 \to X$, where $1 \in S^1 \mapsto x_0 \in X$.

So $\pi_1(S^1, 1) = [(S^1, 1), (S^1, 1)]$. A nice choice of generator is the identity, and any map of degree n realizes the element n, so by this formulation, they must be homotopic.