Topology

Midterm Solutions

1. Classify the letters A, B, C, D, E, F, G up to homeomorphism. That is, divide the letters into groups such that any two in the same group are homeomorphic, and then prove that letters from different groups are *not* homeomorphic. (Think of each letter as a compact subset of $\mathbb{R}^2$, as printed above.)

   Answer. The letters in the groups (A), (B), (C, G), (D), (E, F) are homeomorphic; in fact (C, G) are both homeomorphic to $[0, 1]$, and E is homeomorphic to F after straightening one edge.

   To see letters in different groups are *not* homeomorphic, let $n(X)$ to be the maximum of $|E|$ over all finite sets $E \subset X$ such that $X - E$ is connected, and let $b(X)$ the number of points in X such that $X - \{x\}$ is connected. We find the values:

X	A	B	C	D	E	F	G
n(X)	3	2	2	1	3	3	2
b(X)	∞	∞	2	∞	3	3	2

   (For example, to see $n(\mathsf{A}) = 3$, note that you can remove the endpoints of the feet of A and one point from top, and it's still connected, but if you remove one more point it falls into at least 2 pieces.)

   If X and Y are homeomorphic, then $n(X) = n(Y)$ and $b(X) = b(Y)$. Thus this table shows letters in different groups are not homeomorphic.

2. Let X be a path-connected topological space. Show that $X^{\mathbb{N}}$, in the product topology, is also path-connected.

   Answer. Let $x = (x_i)$ and $y = (y_i)$ be points in $X^{\mathbb{N}}$. Since X is path-connected, there exist continuous maps $f_i : [0, 1] \to X$ with $f_i(0) = x_i$ and $f_i(1) = y_i$. Define $f : [0, 1] \to X^{\mathbb{N}}$ by $f(t) = (f_i(t))$. Since each coordinate f_i of f is continuous, so is f (Munkres, Theorem 19.6). Since $f(0) = x$ and $f(1) = y$, we have shown that $X^{\mathbb{N}}$ is path-connected.

3. Let $f : X \to Y$ be a map between compact Hausdorff spaces. Show that if the graph

   $$\mathrm{gr}(f) = \{(x, y) \in X \times Y \ : \ f(x) = y\}$$

   is a closed subspace of $X \times Y$, then f is continuous.

   Answer. Let F be a closed subset of Y. Then F is compact, and thus $X \times F$ is a compact subspace of $X \times Y$. Since $\mathrm{gr}(f)$ is closed, $(X \times F) \cap \mathrm{gr}(f)$ is also compact. But the projection map $\pi : X \times Y \to X$ is continuous, so

   $$\pi((X \times F) \cap \mathrm{gr}(f)) = f^{-1}(F)$$

   is a compact subset of X. Since X is Hausdorff, $f^{-1}(F)$ is closed. Thus f^{-1} sends closed sets to closed sets, and therefore f is continuous.

4. Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of polynomials such that $\lim f_n(x) = 1$ if $x = 1/2$ and $\lim f_n(x) = 0$ otherwise. Prove that $\sup_n \sup_{x \in [0,1]} |f_n'(x)| = \infty$.

   Answer. We prove the contrapositive. Suppose instead there exists an M such that $|f_n'(x)| \leq M$ for all x and M. Choose $\epsilon < 1/(10M)$ with $0 < \epsilon < 1/2$. Then by the mean-value theorem from calculus, for each n we have

   $$|f_n(1/2) - f_n(1/2 + \epsilon)| \leq M\epsilon \leq 1/10.$$

   Therefore $|\lim f_n(1/2) - f_n(1/2 + \epsilon)| \leq 1/10$ as well, so $\lim f_n(x)$ cannot be 1 for $x = 1/2$ and 0 for $x = 1/2 + \epsilon$.

5. Let $X_1 \supset X_2 \supset X_3 \ldots$ be a nested sequence of closed, connected subsets of a compact Hausdorff space Y.

(a) Show that $X = \bigcap X_i$ is connected.
(b) Given an example showing that X can be disconnected if Y is not compact.

Answer. (a) Suppose the compact set X is not connected. Then we can write it as a disjoint union $X = A \sqcup B$ of two nonempty compact sets. Since Y is Hausdorff, there are disjoint open sets U, V in Y with $A \subset U$ and $B \subset V$ (by homework problem 3.26(5).)

Let $F_i = X_i - (U \cup V)$. This is a nested sequence of closed sets, and $\bigcap F_i = X - (U \cup V) = \emptyset$. Since Y is compact, we must have $F_n = \emptyset$ for some n. But then we have $X_n \subset U \cap V$. Since X is a subset of X_n, both $X_n \cap U$ and $X_n \cap V$ are nonempty, and thus X_n is disconnected. But we have assumed X_n is connected, so X is connected.

(b) Let $X_i \subset \mathbb{R}^2$ be the 'infinite ladder' obtained by adjoining to the lines $x = 0$ and $x = 1$ the rungs $[0,1] \times \{i, i+1, i+2, \ldots\}$. Then each X_i is closed and connected, but $X = \bigcap X_i$ has no rungs left — it consists of just the lines $x = 0$ and $x = 1$, so it is disconnected.

6. Show that $\mathbb{R}_\ell$, the real numbers with the lower limit topology, is not metrizable.

   Answer 1. The space $\mathbb{R}_\ell$ is separable ($\mathbb{Q}$ is dense) but not second countable (for each $a \in \mathbb{R}$, there must be a basis element with $a \in B_a \subset [a, a+1)$; since $a = \inf B_a$, the map $a \mapsto B_a$ is $1-1$, so there are at least as many basis elements as there are points $a \in \mathbb{R}$). But for a metric space, second countability and separability are equivalent (a result presented in class; see also Munkres, Exercise 3.31(5).) Therefore $\mathbb{R}_\ell$ is not metrizable.

   Answer 2. Suppose $\mathbb{R}_\ell$ is metrizable, with metric d. Let X_n be the set of $a \in [0,1]$ such that $B(a, 1/n) \subset [a, \infty)$, where $B(\cdot)$ denotes a ball in the d-metric. Since $[a, \infty)$ is open in $\mathbb{R}_\ell$, we have $[0,1] = \bigcup X_n$. Thus X_n is uncountable for some n. Let $x \in [0,1)$ be the largest value such that $[0,x] \cap X_n$ is countable. Then $(x, x+\epsilon) \cap X_n$ is uncountable for every $\epsilon > 0$, so there exist $x_i \in X_n$ converging to x from above. Then $x_i \to x$ in $\mathbb{R}_\ell$, so $d(x_i, x) \to 0$, and thus $d(x_i, x) < 1/n$ for some i. But then $B(x_i, 1/n)$ contains $x < x_i$, contrary to the assumption that x_i belongs to X_n. Therefore $\mathbb{R}_\ell$ is not metrizable.

   Answer 3. If $\mathbb{R}_\ell$ is metrizable, then so is $\mathbb{R}_\ell^2$. But $\mathbb{R}_\ell^2$ is not normal (as shown in Munkres) so neither $\mathbb{R}_\ell^2$ nor $\mathbb{R}_\ell$ is metrizable.

7. Let (X, d) be a nonempty compact metric space, and let $f : X \to X$ be a continuous map such that whenever $x \neq y$ we have $d(f(x), f(y)) < d(x, y)$. Show that f has a unique fixed-point (i.e. there is a unique $x \in X$ such that $f(x) = x$).

   Answer. Let $h(x) = d(x, f(x))$. Then $h : X \to \mathbb{R}$ is a continuous function on a compact space, so it achieves its minimum; that is, there is an $x_0 \in X$ such that $h(x_0) \leq h(x)$ for all $x \in X$.

   We claim $f(x_0) = x_0$. If not, then we have

$$h(f(x_0)) = d(f(x_0), f(f(x_0))) < d(f(x_0), x_0) = h(x_0),$$

   contradicting the fact that $h(x_0)$ is the minimum of h.

   To see x_0 is the unique fixed-point of f, suppose $x \neq x_0$. Then $d(f(x), f(x_0)) = d(x, f(x_0)) < d(x, x_0)$, so $f(x) \neq x$.