Topology
Take-home Midterm, 9 March 2001
Math 131 – Harvard University – Spring 2001
Due in class on Wednesday, 14 March 2001

Name ______________________________

Do any 6 of the following 7 problems.
Aim for clear, concise, complete answers. Refer only to Munkres and your class notes. Do not collaborate; all work must be done on your own. Each problem is worth 15 points.

1. Classify the letters A, B, C, D, E, F, G up to homeomorphism. That is, divide the letters into groups such that any two in the same group are homeomorphic, and then prove that letters from different groups are *not* homeomorphic. (Think of each letter as a compact subset of $\mathbb{R}^2$, as printed above.)

2. Let X be a path-connected topological space. Show that $X^{\mathbb{N}}$, in the product topology, is also path-connected.

3. Let $f : X \to Y$ be a map between compact Hausdorff spaces. Show that if the graph
$$\mathrm{gr}(f) = \{(x, y) \in X \times Y \ : \ f(x) = y\}$$
is a closed subspace of $X \times Y$, then f is continuous.

4. Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of polynomials such that $\lim f_n(x) = 1$ if $x = 1/2$ and $\lim f_n(x) = 0$ otherwise. Prove that $\sup_n \sup_{x\in[0,1]} |f_n'(x)| = \infty$.

5. Let $X_1 \supset X_2 \supset X_3 \ldots$ be a nested sequence of closed, connected subsets of a compact Hausdorff space Y.

   (a) Show that $X = \bigcap X_i$ is connected.
   (b) Given an example showing that X can be disconnected if Y is not compact.

6. Show that $\mathbb{R}_\ell$, the real numbers with the lower limit topology, is not metrizable.

7. Let (X, d) be a nonempty compact metric space, and let $f : X \to X$ be a continuous map such that whenever $x \neq y$ we have $d(f(x), f(y)) < d(x, y)$. Show that f has a unique fixed-point (i.e. there is a unique $x \in X$ such that $f(x) = x$).