

Topology Final – Solutions

Math 131 – Harvard University – Spring 2001

1. Let X be a compact space, and $f_n : X \rightarrow \mathbb{R}$ a sequence of continuous functions such that $f(x) = \lim f_n(x)$ exists for each $x \in X$. Show that $f_n \rightarrow f$ uniformly if and only if the family of functions $\mathcal{F} = \{f_1, f_2, \dots\}$ is equicontinuous.

A. If $f_n \rightarrow f$ uniformly then the countable set $\{f, f_1, f_2, \dots\}$ is a compact subset of $C(X)$, so \mathcal{F} is equicontinuous.

Conversely, if \mathcal{F} is equicontinuous then given $\epsilon > 0$ there is a finite cover of X by open sets U_i such that $\text{diam } f_n(U_i) < \epsilon$ for all n . Passing to a limit, we find the same is true for $f(U_i)$. Now choose points $x_i \in U_i$ and an N such that $|f(x_i) - f_n(x_i)| < \epsilon$ for all $n > N$. Then $|f(x) - f_n(x)| < 3\epsilon$ for all $x \in X$ and $n > N$, and thus $f_n \rightarrow f$ uniformly.

2. Let $\{U_1, \dots, U_n\}$ be an open cover of a compact Hausdorff space, and let $f_{ij} : U_i \cap U_j \rightarrow \mathbb{R}$ be continuous functions satisfying $f_{ij} = -f_{ji}$ and $f_{ij} + f_{jk} = f_{ik}$ on $U_i \cap U_j \cap U_k$.

Show there exist continuous functions $g_i : U_i \rightarrow \mathbb{R}$ such that $g_i - g_j = f_{ij}$ on $U_i \cap U_j$.

A. Let ρ_i be a partition of unity subordinate to the cover U_i , and set $g_i = \sum \rho_k f_{ik}$, where we extend $\rho_k f_{ik}$ from $U_i \cap U_k$ to U_i by zero. Then we have:

$$g_i - g_j = \sum \rho_k (f_{ik} - f_{jk}) = \sum \rho_k (f_{ik} + f_{kj}) = \sum \rho_k f_{ij} = f_{ij}.$$

3. Let $f : X \rightarrow Y$ be a surjective local homeomorphism between compact Hausdorff spaces. Prove that f is a covering map.

A. By compactness there is a covering of X by finitely many open sets, each of which maps 1-1 to Y . Thus f is finite-to-one. Given $y \in Y$, let $f^{-1}(y) = \{y_1, \dots, y_n\}$ and choose disjoint neighborhoods U_i of y_i (possible since X is normal). Choose neighborhoods $V_i \subset U_i$ of y_i such that $f|_{V_i}$ is a homeomorphism. Finally set $W = \bigcap f(V_i)$ and $W_i = V_i \cap f^{-1}(W)$. Then $f(W_i) = W$ and $f|_{W_i}$ is a homeomorphism, so W is evenly covered and thus f is a covering map.

4. Prove there is no retraction of the Möbius band onto its boundary. (See Munkres p. 450 for a picture of the Möbius band.)

A. Let $B \cong S^1$ be the boundary of the Möbius band M , and let $i : B \rightarrow M$ be the inclusion. Then $\pi_1(B) \cong \pi_1(M) \cong \mathbb{Z}$ and $i_*(n) = 2n$. If there is a retraction $r : M \rightarrow B$ then we have $r_*(i_*(n)) = n$. But there is no homomorphism $r_* : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $r_*(2n) = n$.

5. Compute the fundamental group of $S^3 - S^1$. Here

$$S^3 = \{p \in \mathbb{R}^4 : |p| = 1\}, \quad \text{and} \quad S^1 = \{p = (p_1, p_2, 0, 0) : p_1^2 + p_2^2 = 1\}.$$

A. It is easy to see that $\mathbb{R}^4 - \mathbb{R}^2$ deformation retracts radially to $S^3 - S^1$, so they have the same fundamental group. But it is also easy to deformation retract $\mathbb{R}^4 - \mathbb{R}^2$ to $\mathbb{R}^2 - \mathbb{R}^0$ and then to S^1 . Thus $\pi_1(S^3 - S^1) \cong \pi_1(S^1) \cong \mathbb{Z}$.

6. Let $A_{ij} > 0$ be a positive 3×3 matrix. Using topology, prove that A has a positive eigenvector; that is, prove there exists $v_i > 0$ such that $Av = \lambda v$ for some $\lambda > 0$.

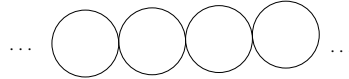
A. The region $B = \{v : v_i \geq 0, |v| = 1\} \subset \mathbb{R}^3$ is homeomorphic to B^2 , so the map $f : B \rightarrow B$ given by $f(v) = Av/|v|$ has a fixed-point $v \in B$. Then $Av = \lambda v$ with $\lambda > 0$. Since $|v| = 1$ and $A_{ij} > 0$, we have $(Av)_i > 0$ for all i ; therefore $v_i > 0$ for all i .

7. Let $[f] \in \pi_1(X, x_0)$ be represented by a path $f : [0, 1] \rightarrow X$ that begins and ends at x_0 . Let $g : [0, 1] \rightarrow [0, 1]$ be a homeomorphism. Prove that $[f \circ g] = [f]$ or $[f]^{-1}$ in $\pi_1(X, x_0)$.

A. It is straightforward to see that g permutes the endpoints $\{0, 1\}$ of $[0, 1]$. If $g(0) = 0$ and $g(1) = 1$, then $f_t(x) = f(tx + (1-t)g(x))$ gives a homotopy between $f_0 = f \circ g$ and $f_1 = f$ rel endpoints. Thus $[f \circ g] = [f]$. Similarly, if g exchanges 0 and 1, then the homotopy $f_t(x) = f(t(1-x) + (1-t)g(x))$ shows $[f \circ g] = [f]^{-1}$.

8. Compute the fundamental group of $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$. (Here \mathbb{RP}^2 is the real projective plane.) Draw a picture of the universal cover of X .

A. The fundamental group is $\mathbb{Z}/2 * \mathbb{Z}/2$ by Seifert-van Kampen. The universal cover is an infinite necklace of 2-spheres, shown below.



9. Let $V : S^2 \rightarrow \mathbb{R}^3$ be a continuous vector field tangent to sphere (so $V(p) \cdot p = 0$ for every $p \in S^2$). Show that V must vanish at some point.

A. For p in the equator S^1 let $\theta(p)$ be the angle between $V(p)$ and the equator. Regarding the angle as ranging in $\mathbb{R}/2\pi\mathbb{Z} \cong S^1$, we obtain a continuous map $\theta : S^1 \rightarrow S^1$. Flattening out the northern hemisphere to a 2-ball, and using the fact that V gives a nowhere-zero vector field on the ball, we conclude that the degree of θ is -1 . Flattening out the southern hemisphere, we similarly conclude that the degree of θ is 1. Thus no such vector field exists.

10. Let $X \subset \mathbb{R}^2$ be the compact set consisting of all line segments joining $p = (0, 1)$ to $q_n = (1/n, 0)$, $n = 1, 2, 3, \dots$, together with the segment from p to $(0, 0)$. Let $A = \{(0, 0)\} \subset X$.



- (i) Show that the inclusion $A \subset X$ is a homotopy equivalence.
(ii) Show that A is not a deformation retract of X .

A. (i) Let $i : A \rightarrow X$ be the inclusion and $j : X \rightarrow A$ be the constant map. Then $j \circ i = \text{id}_A$. We must show $i \circ j = j$ is homotopic to id_X . Since X is path-connected, j is homotopic to the constant map to the p of X . But the identity map from X to itself is homotopic to the constant map to p by a radial homotopy. Thus i is a homotopy equivalence.

(ii) Now suppose A is a deformation retract of X . Then there is a homotopy $h : X \times I \rightarrow X$ such that $h(x, t) = x$, $h(x, 1) = (0, 0)$ and $h((0, 0), t) = (0, 0)$ for all t . Since h sends $A \times I$ to A , there is a neighborhood U of A in X such that h sends $U \times I$ into $B(A, 1/2)$; in particular, such that $h(x, t) \neq (1, 0)$ for all $x \in U$ and for all t . But U contains $q_n = (1/n, 0)$ for some $n > 0$, and as t ranges from 0 to 1, $h(q_n, t)$ gives a path from q_n to $(0, 0)$ in X . Any such path must pass through $(1, 0)$, contrary to the fact that $h(q_n, t)$ remains in $B(A, 1/2)$.

Thus A is not a deformation retract of X .