

Topology Final

Math 131 – Harvard University – Spring 2001

Due Friday, 11 May 2001, 12:00 pm

Please hand in your completed final to the staff in room 325 Science Center.

Name:

Aim for *concise*, clear answers. Refer only to Munkres and your class notes. Do not collaborate; all work must be done on your own.

Do any 9 of the following 10 problems. All problems have equal weight.

1. Let X be a compact space, and $f_n : X \rightarrow \mathbb{R}$ a sequence of continuous functions such that $f(x) = \lim f_n(x)$ exists for each $x \in X$. Show that $f_n \rightarrow f$ uniformly if and only if the family of functions $\mathcal{F} = \{f_1, f_2, \dots\}$ is equicontinuous.
2. Let $\{U_1, \dots, U_n\}$ be an open cover of a compact Hausdorff space, and let $f_{ij} : U_i \cap U_j \rightarrow \mathbb{R}$ be continuous functions satisfying $f_{ij} = -f_{ji}$ and $f_{ij} + f_{jk} = f_{ik}$ on $U_i \cap U_j \cap U_k$. Show there exist continuous functions $g_i : U_i \rightarrow \mathbb{R}$ such that $g_i - g_j = f_{ij}$ on $U_i \cap U_j$.
3. Let $f : X \rightarrow Y$ be a surjective local homeomorphism between compact Hausdorff spaces. Prove that f is a covering map.
4. Prove there is no retraction of the Möbius band onto its boundary. (See Munkres p. 450 for a picture of the Möbius band.)
5. Compute the fundamental group of $S^3 - S^1$. Here

$$S^3 = \{p \in \mathbb{R}^4 : |p| = 1\}, \quad \text{and} \quad S^1 = \{p = (p_1, p_2, 0, 0) : p_1^2 + p_2^2 = 1\}.$$

6. Let $A_{ij} > 0$ be a positive 3×3 matrix. Using topology, prove that A has a positive eigenvector; that is, prove there exists $v_i > 0$ such that $Av = \lambda v$ for some $\lambda > 0$.
7. Let $[f] \in \pi_1(X, x_0)$ be represented by a path $f : [0, 1] \rightarrow X$ that begins and ends at x_0 . Let $g : [0, 1] \rightarrow [0, 1]$ be a homeomorphism. Prove that $[f \circ g] = [f]$ or $[f]^{-1}$ in $\pi_1(X, x_0)$.
8. Compute the fundamental group of $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$. (Here \mathbb{RP}^2 is the real projective plane.) Draw a picture of the universal cover of X .
9. Let $V : S^2 \rightarrow \mathbb{R}^3$ be a continuous vector field tangent to sphere (so $V(p) \cdot p = 0$ for every $p \in S^2$). Show that V must vanish at some point.
(Hint: when does a vector field defined on the equator have a nowhere-vanishing extension to the northern hemisphere?)
10. Let $X \subset \mathbb{R}^2$ be the compact set consisting of all line segments joining $p = (0, 1)$ to $q_n = (1/n, 0)$, $n = 1, 2, 3, \dots$, together with the segment from p to $(0, 0)$. Let $A = \{(0, 0)\} \subset X$.



- (i) Show that the inclusion $A \subset X$ is a homotopy equivalence.
- (ii) Show that A is not a deformation retract of X .