MIDTERM

Algebra — Math 122 — Fall 2002

1. Let D_n be the dihedral group generated by a reflection r and a rotation ρ of order n. Express $g = r\rho^{n-2}r\rho^5$ in the form $\rho^i r^j$.

   Answer. In any dihedral group we have $r\rho = \rho^{-1}r$. Thus $g = r^2\rho^2\rho^5 = \rho^7 = \rho^7 r^0$.

2. Let $V = \{a_0x^n + a_1x^{n-1} + \cdots + a_n \ : \ a_i \in \mathbb{R}\}$ be the vector space of real polynomials of degree n. Define $A : V \to V$ by

$$A(p) = p''(x) - p'(x) + p(x).$$

   What is the dimension of the image of A? Explain your answer.

   Answer. The kernel of A is clearly 0, since the leading coefficient of $A(p)$ is the same as the leading coefficient of p. Thus the dimension of the image of A is the same as the dimension of V, which is $n + 1$.

3. Let g and h be rigid motions of $\mathbb{R}^2$ preserving orientation.

   (a) Show that $t = ghg^{-1}h^{-1}$ is a translation (or the identity).

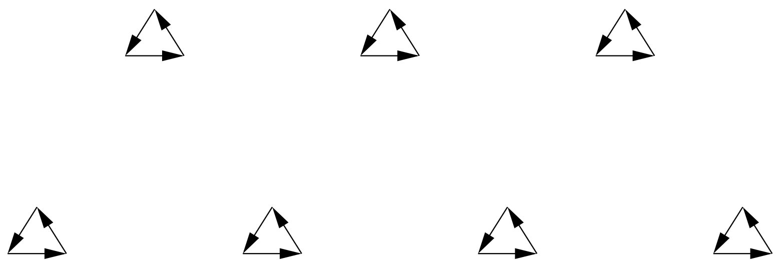
   (b) Show that t need not be the identity if g and h are reflections.

   Answer. (a) The derivative map $D : M \to O_2(\mathbb{R})$ is a homomorphism, so $Dt = (Dg)(Dh)(Dg)^{-1}(Dh)^{-1}$. Since g and h preserve orientations, their derivatives lie in the abelian group $SO_2(\mathbb{R})$. Thus $Dt = I$ and therefore t is a translation.

   (b) Consider the pair of reflections $g = r$ and $h = \rho r \rho^{-1}$ in the dihedral group $D_n \subset O_2(\mathbb{R})$. Then $g = g^{-1}$ and $h = h^{-1}$, so $t = (gh)^2 = (r\rho r\rho^{-1})^2 = \rho^4$. Thus if $n = 3$ or $n > 4$, we have $t \neq \mathrm{id}$.

4. Draw a picture of a wallpaper pattern in $\mathbb{R}^2$ whose symmetry group contains an element of order 3, but does *not* contain any orientation-reversing element.

Answer. A typical solution is below. Since the arrows run in a counter-clockwise direction, the pattern is not equivalent to its mirror image.



5. Mark each of the following assertions True (T) or False (F).

(a) T. The dihedral group D_2 is abelian. (We have $r\rho = \rho^{-1}r = \rho r$ since $\rho^2 = 1$.)

(b) T. If H_1 and H_2 are normal subgroups of G, then so is $H_1 \cap H_2$.

(c) F. If $\det(tI - A) = (t-1)^2$, then A is not diagonalizable. (A might be the identity matrix.)

(d) F. We have $3^{34} = 1 \bmod 17$. (By Fermat's little theorem, $a^p = a \bmod p$. Thus $3^{34} = 3^2 = 9 \neq 1 \bmod 17$.)

(e) F. If F is an arbitrary field, then F^* is a cyclic group. (Consider for example $F = \mathbb{Q}$.)

(f) T. If $A \in \mathrm{O}_2(\mathbb{R})$ has order 4, then $\det(A) = 1$. (If $\det(A) = -1$ then A is a reflection, so A has order 2.)

(g) T. There exists a finite subgroup $G \subset \mathrm{SO}_3(\mathbb{R})$ of any given order. (The group generated by a single rotation through angle $2\pi/n$ has order n.)

(h) T. The group $\mathrm{SO}_2(\mathbb{R})$ is abelian. (Any two rotations of the plane commute, since $\rho_\alpha \rho_\beta = \rho_{\alpha+\beta}$.)