Math 122, Solution Set No. 7

1 6.1.1

This rule does define a (left) operation of G on itself. $1 \cdot x = x(1^{-1}) = x$, and $(gh) \cdot x = x(gh)^{-1} = xh^{-1}g^{-1} = g \cdot (h \cdot x)$. (Here $\cdot$ denotes the group action).

2 6.1.4

Let G be a p-group acting on S such that $p \nmid |S|$. Assume there is no fixed point. Then for each $s \in S$, $\text{Stab}(s) \neq G \Rightarrow |O_s| = p^e, e \geq 1$ by the Counting Formula. Then since we can write $S = \sum_{s \in S} O_s$, and $p | O_s \forall s \in S$, we have $p|S$, a contradiction. Therefore there is a fixed point for the action of G on S.

3 6.1.5

Note: since any rigid motion can be written as the composition of translations, rotations, and reflections, it suffices to consider conjugating by these elements.

Consider translations: Conjugating t_a by r gives $t_{r(a)}$; conjugating by ρ_θ gives $t_{\rho_\theta(a)}$; and conjugating by another t_b gives t_a. So if we conjugate by any rigid motion of the plane, t_a will be conjugate to a translation by the same distance. Also, t_a is conjugate to any translation of the same magnitude, so all translations of the same magnitude form a conjugacy class (the identity is translation by the zero vector).

Consider rotations: Conjugating ρ_θ by another rotation gives ρ_θ; by a translation t_a gives rotation of angle θ around the point a, and conjugating by r gives $\rho_{-\theta}$. So the conjugacy class of a rotation is rotations by $\pm\theta$ around any point in the plane.

Consider reflections: if we conjugate r by the motion $t_a\rho_\theta$, we obtain $t_{a+\rho_{2\theta}(r(-a))}\rho_{2\theta}r$. Let $\varphi = 2\theta$. Also note that since rigid motions are bijections of the plane, by suitably choosing a we can obtain $b = a + \rho_{2\theta}(r(-a))$ for any $b \in \mathbb{R}^2$. Therefore r is conjugate to any orientation-reversing motion, including glide reflections of all magnitudes. Note that since conjugacy is an equivalence relation, all of the orientation-reversing elements of M are in the same conjugacy class.

4 6.1.6

(a) $1+1+1+2+5$. This is not a valid class equation, since it implies that $|Z(G)| = 3 \nmid 10$, a contradiction since $Z(G) \leq G$. (b) $1+2+2+5$ is the class equation for D_5. (c) $1+2+3+4$ is not valid, since it implies $\exists s$ such that $|O_s| = 3 \nmid 10$, a contradiction. (d) $1+1+2+2+2+2$. This is not a valid class equation. $|O_s| = 2 \Rightarrow |G_s| = 5$. But $Z(G) \leq G_s$ and $|Z(G)| = 2 \nmid 5$, a contradiction.

5 6.1.8

(a) Clearly $1, -1$ are in the center of Q_8 since they are the only elements of order 2. Furthermore, it can be checked by computation that $C_i = \{\pm i\}$ and likewise for j, k. Therefore the class equation is $10 = 1+1+2+2+2$.

(c) $10 = 1+2+2+5$, a special case of part (e)

(e) Consider $D_n = \langle x, y | x^n = y^2 = 1 \text{ and } xy = yx^{-1} \rangle$. Clearly the identity is in its own conjugacy class, and if we conjugate an element x^j by x^m, we get x^j, while if we conjugate by $x^m r$ we get x^{-j}. Now consider $x^j x^m r x^{-j} = x^j r x^m r r x^{-j} = x^{2m} x^{-j} r$. If n is odd, then x^2m cycles through all elements x^j as m does, and so all elements $x^j r$ form a single conjugacy class. If n is even, x^2m cycles through all elements x^j, j even as m does, and it is easily seen that all elements $x^j r$ with j

odd also form a conjugacy class (also note that in this case, $x^{n/2}$ is its own inverse). So the class equations are:

n even: $2n = 1 + 1 + 2 + \cdots + 2 + n/2 + n/2$

n odd: $2n = 1 + 2 + \cdots + 2 + n$

(g) While it is possible to divide the matrices into categories based on their trace, the quickest way to do this problem is to compute the conjugacy classes by direct computation (using, for example, mathmatica). The result is that all trace 0 matrices are conjugate, while there are three conjugacy classes of order 4, 4, and 1 of the matrices of both trace 1 and trace 2. The class equation for $SL_2(\mathbb{F}_3)$ is then $24 = 1 + 1 + 4 + 4 + 4 + 4 + 6$. Note furthermore that $SL_2(\mathbb{F}_3)$ is NOT isomorphic to $A_4 \times \mathbb{Z}/2\mathbb{Z}$.

6 6.1.12

Let N be normal in G and $|G|$ odd. Since N is normal, it is a union of conjugacy classes contained in N; the orders of these classes must be odd, since they necessarily divide G. So since the identity is in its own conjugacy class, the only possible class equations are $5 = 1+1+1+1+1$ and $5 = 1+1+3$. The second implies that $|Z(G) \cap N| = 2$, a contradiction since $Z(G) \cap N$ is a subgroup of G and so by Lagrange's Theorem, its order is necessarily odd. Therefore every element of N is in its own conjugacy class, i.e. $N \leq Z(G)$.