122 Solution Set 6

1 5.5.8

(a) $g(0) = 0$ for all $g \in G$. Thus $0_0 = \{0\}$. Let e_1 be the first vector of the standard basis for $\mathbb{R}^n$ and $v \in \mathbb{R}^n$ be an arbitrary vector, not equal to zero. We may construct an element $h \in G$ such that $he_1 = v$ by simply letting the first column of the matrix of h be v (with respect to the standard basis. Thus there are two orbits; we have $\mathbb{R}^n = O_0 \cup O_{e_1}$. As a check, note that $gv = 0$ implies $v = 0$ for $g \in G$.

(b) Write $g \in G$ with respect to the standard basis, in terms of its column vectors $v_1, ..., v_n$ as

$$g = (v_1, ..., v_n).$$

We have $ge_1 = v_1$. Thus, the stabilizer of e_1 is all matrices of the form $g = (e_1, v_2, ..., v_n) \in G$.

2 5.6.1

$gaH = aH \iff gah = ah'$ (for some $h, h' \in H$) $\iff ga = ah''$ (for some $h \in H$) $\iff g = ah''a^{-1} \iff g \in aHa^{-1}$. Thus the stabilizer is aHa^{-1}.

3 5.6.4

(a) The stabilizer, by definition, is all permutations of $\{1, ..., n\}$ that leave 1 fixed, that is, it is all permutations of $\{2, ..., n\}$, which is clearly isomorphic to S_{n-1}.

(b) S_n is generated by the all of the transpositions of the n elements. The only transpositions that are not in $H = Stab(\{1\})$ are $(12), (13), ...(1n)$. Thus, $S = H \cup \{\cup_{1<j\leq n}(1j)H\}$.

I claim that these cosets of H are distinct. By the counting formula, we have $|S_n| = n! = |O_{\{1\}}||H| = |0_{\{1\}}|(n-1)!$ which implies $|O_{\{1\}}| = n$. By Lagrange, $|S_n| = |H|[G : H]$, so we have $|O_{\{1\}}| = [G : H] = n$. If these cosets were not distinct, we would have less than n cosets of H in G, which contradicts the fact that $[G : H] = n$.

(c) We have $\varphi((1j)H) = (1j) \cdot 1 = j$.

4 5.7.2

As stated in the problem (and as you can check by drawing it), there are exactly two ways to inscribe a regular tetrahedron in a cube. If we denote one such tetrahedron by Δ, we thus have $|O_\Delta| = 2$. From 5.9.1, the order of rotational symmetries of a cube is 24. By the counting formula, we have $|Stab(\Delta)| = 24/2 = 12$.

5 5.7.5 (thanks to Jason Adaska)

First we assume that all indices $[G : K], [H : K], [G : H]$ are finite, otherwise this equation doesn't make sense. Consider the action of G on G/K. The stabilizer of K for this action is K, so we have $|O_K| = [G : K]$ by the counting formula. Now consider the orbit of K under the action of H on G/K, denote this as $|O'_K|$. Again, the stabilizer is K, so we have $|O'_K| = [H : K]$. Let $a_1, ..., a_n$ be elements of G such that $G = \cup_{1 \le i \le n} a_i H$ is a disjoint union. Thus we have

$$\begin{aligned} O_K &= \{a_i HK : 1 \le i \le n\} \\ &= \{a\{hK : h \in H\} : 1 \le i \le n\} \\ &= \{a_i O'_K : 1 \le i \le n\}. \end{aligned}$$

Thus $|O_s| = |G/H||O'_H|$, which implies $[G : K] = [G : H][H : K]$.

6 5.8.5

Let G be the group acting on S. There are 2! possible permutations of the elements of the orbit with order two, and 3! possible permutations of the elements of the orbit order three. These correspond to all of the possible actions of a group on this set; any action just permutes the elements in some way. Because the action is faithful, no two elements of G correspond to the same action. Thus $|G| \le 12$.

Suppose $|G| = 12$. G must have elements corresponding to all of the permutations of S, namely $e, (12), (13), (23), (123), (132), (45)$ or any combination of these elements. Thus G is completely defined by its action on S, and we see that G is isomorphic to $S_3 \times S_2$. Without loss of generality, let the two orbits be $O_3 = \{1, 2, 3\}$ and $O_2 = \{4, 5\}$. We let the nontrivial element of S_2 transpose $4, 5$ and the elements of S_3 act as usual on $\{1, 2, 3\}$.

Suppose $|G| \ne 12$. Then, since $|O_3|||G|$ and $|O_2|||G|$, we have $|G| = 6$. From a previous problem set, the only groups of order 6 are C_6 and S_3.

C_6: We have that C_6 is isomorphic to $C_3 \times C_2$; if $< x >= C_3$ and $< y >= C_2$, we define an action by sending x to the permutation (123) and y to the permutation (12).

S_3: We define a homomorphism $S_3 \to S_3 \times C_2$ by $\sigma \to \sigma \times sgn(\sigma)$. Then let S_3 act as usual on $\{1, 2, 3\}$ and C_2 as before on $\{4, 5\}$.

7 5.8.7

The subspaces are:

$$1 := \{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}\}$$

$$2 := \{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\}$$

$$3 := \{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\}$$

$$4 := \{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\}$$

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By Prop. 5.8.2, to show that we have a homomorphism from $GL_2(F) \to S_4$, it is enough to show that we have an action of $GL_2(F)$ on this subspace. This is trivial (and we're given this fact in the statement of the problem). The kernel of this map is elements of $g \in GL_2(F)$ such that $g \cdot 1 = 1, g \cdot 2 = 2, g \cdot 3 = 3, g \cdot 4 = 4$. But the only such g are scalar matrices. There are 2 in $GL_2(F)$, namely $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ and $\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$. Now, $|GL_2(F)| = (3^2 - 1)(3^2 - 3) = 48$, so $|im(\varphi)| = |GL_2(F)|/|ker(\varphi)| = 24$. But $|S_4| = 24$, so $im(\varphi) = S_4$.

8 5.9.1

I'm just going to list some answers; most didn't have any trouble with the explanations.

Octahedron: Edge pole orbit order-6. Face pole orbit order-4. Vertex pole orbit order-3.

Icosahedron: Edge pole orbit order-15. Face pole orbit order-10. Vertex pole orbit order 6.

9 5.9.3

There are 4 diagonals, all of which are in the same orbit under rotations. Thus $H := |Stab(diagonal)| = 24/4 = 6$ by the counting formula, which implies that $Stab(diagonal)$ is isomorphic to S_3 or $C_3 \times C_2$. Thinking geometrically, we see that H is generated by two rotations, each of order three, about the vertices defining the diagonal, and a symmetry of order two which flips the two vertices defining the diagonal. Further, if we conjugate the flip by a rotation, we get another rotation, not a flip. Thus the subgroup of H of order two, generated by this flip, is not normal. This rules out the possibility that H is isomorphic to $C_3 \times C_2$, for every subgroup of order two in $C_3 \times C_2$ is normal. Thus H is isomorphic to S_3.

10 5.9.5

Let S be the set of pairs of opposite vertices of the icosahedron. Then S has order 6. Further, any pair can be taken to any other pair via some element of the icosahedral group. Pick some pair p. The order of the icosahedral group is 60 (from Artin), so we have $60 = |O_p||Stab(p)| = 6|Stab(p)|$. Thus $|Stab(p)| = 6$, and $Stab(p)$ is a subgroup of the icosahedral group.