Math 122, Solution Set No. 11

1 7.2.10

Note that $M_n(\mathbb{R}) \equiv \mathbb{R}^{n^2}$ under the identification $[A_1, \ldots, A_n] \leftrightarrow ((A_1)^t, \ldots, (A_n)^t)$. Using this identification, we see that the given form is just the standard dot product on $\mathbb{R}^{n^2}$ and therefore positive-definite, with an orthonormal basis given by the matrices $e_{ij}, 1 \leq i, j, \leq n$ where e_{ij} is the $n \times n$ matrix having a 1 in the $(i, j)^{\text{th}}$ position and zeroes elsewhere.

2 7.2.15

(a) Let $w \in (W_1 + W_2)^\perp \Rightarrow \langle w, w_1 + 0\rangle = 0 \forall w_1 \in W_1 \Rightarrow w \in W_1^\perp$. Likewise $w \in W_2^\perp$ and so $(W_1 + W_2)^\perp \subseteq W_1^\perp \cup W_2^\perp$. If $w \in W_1^\perp \cup W_2^\perp$, then it follows from bilinearity that $w \in (W_1 + W_2)^\perp$. Therefore $(W_1 + W_2)^\perp = W_1^\perp \cup W_2^\perp$.

(b) Let $w \in W$. Then for any $v \in W^\perp$, $\langle v, w\rangle = 0$ and so $w \in W^{\perp\perp} \Rightarrow W \subseteq W^{\perp\perp}$.

(c) Let $W_1 \subseteq W_2$ and $w \in W_2^\perp$. Then $\langle w, w_2\rangle = 0 \forall w_2 \in W_2 \Rightarrow \langle w, w_1\rangle = 0 \forall w_1 \in W_1$ since $W_1 \subseteq W_2$. Therefore $W_2^\perp \subseteq W_1^\perp$.

3 7.2.17

(a) It is easy to compute that the matrix of the form with respect to the standard basis is:

$$M_{e_{ij}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b, c) It is easy to check that the basis m_{ij} is orthonormal:

$$(m_{11}, m_{12}, m_{21}, m_{22}) = (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{2}/2 \\ -\sqrt{2}/2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$$

and that the signature with respect to this basis is (3, 1).

(d) Likewise check that $\{C_i\}$ is a basis for the trace-zero subspace, where:

$$(c_1, c_2, c_3) = (\begin{bmatrix} \sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{2}/2 \\ -\sqrt{2}/2 & 0 \end{bmatrix})$$

Then the matrix of the form with respect to this basis is computed to be:

$$M_{e_{ij}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Thus the signature is (2, 1).

4 7.2.23

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(a) Let $P \in GL_2(\mathbb{F}_2)$ and consider PAP^t. If we let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $PAP^t = \begin{bmatrix} 2ab & ad+bc \\ ad+bc & 2ab \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ b/c det $(P) = 1$ and $2 \equiv 0$ in $\mathbb{F}_2$. Therefore A is not diagonalizable as the matrix of a bilinear form.

(b) By explicit computation, we can check that the orbits are:

$$\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\},$$

$$\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\},$$

$$\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\},$$

$$\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\},$$

$$\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\},$$

$$\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\},$$