122 Solution Set 10

1 6.7.1

The free group on two generators is nonabelian, whereas the product of two infinite cyclic groups is. Therefore, the two are not isomorphic.

2 6.7.3

Let S be the underlying set of generators for the free group. Further, let $Cl(S)$ be the set of closed words in S, and $Con(S)$ be the set of conjugacy classes of words in S. We define a map $f : Con(S) \to Cl(S)$ by joining the ends of any element in a given conjugacy class. To check that this map is well defined, suppose that x, y are in the same conjugacy class. Then $wxw^{-1} = y$ for some word $w \in S$. Clearly closing y is equivalent to closing x; the two extra words w just cancel out.

Surjective: Given any closed word, break it at an arbitrary point, and consider the conjugacy class of the word that arises. This conjugacy class will clearly map to the given closed word.

Injective: Suppose $x, y \in S$ are two reduced words such that $f(x) = f(y)$, where we think of x, y as representative elements for two conjugacy classes. The only sort of cancellation that can occur when we close x or y is of the form $x = w^{-1}zw$ being sent to the closed word formed from z. It follows that x, y are conjugate.

3 6.8.2

If a normal subgroup contains S, then it will contain all gsg^{-1} for all $g \in G$, $s \in S$. The subgroup generated by the given set, by definition, is the smallest subgroup containing these elements.

4 6.8.10

Let $t_a\rho_\theta r^i, t_b\rho_\varphi r^j$, where $i, j = 0$ or 1, be arbitrary elements of M (we know any element of M can be written in this form from §5.2). We have, using the formulae in §5.2:

$$t_a\rho_\theta r^i t_b\rho_\varphi r^j (t_a\rho_\theta r^i)^{-1}(t_b\rho_\varphi r^j)^{-1} =$$

$$t_a\rho_\theta t_{r(b)}\rho_{-\varphi} r^i r^j r^i r^j (t_a\rho_\theta)^{-1}(t_b\rho_\varphi)^{-1} =$$

$$t_a\rho_\theta t_{r(b)}\rho_{-\varphi}(t_a\rho_\theta)^{-1}(t_b\rho_\varphi)^{-1} =$$

$$t_a t_{\rho_\theta(r(b))}\rho_\theta\rho_{-\varphi}\rho_{-\theta}t_{-a}\rho_{-\varphi}t_{-b} =$$

$$t_{a+\rho_\theta(r(b))+\rho_{-\varphi}(-a)+\rho_{-2\varphi}(-b)}\rho_{-2\varphi}$$

If we vary a, b, φ, θ, we get all of SM.

5 6.8.16

Prove. Let $|G| = n$. We may choose a finite number of generators; just take all the elements of the group as generators (there's only n).

Note that the multiplication table for G completely defines the law of composition in G. Any entry in the table, say $xy = z$, can equivalently be written as the relation xyz^{-1}. The set of relations formed in this manner from the group table is clearly a defining set of relations for the group, and there are only n^2 entries. This provides a finite set of generators and relations that present G.

6 7.1.1

Let $A = [a_{ij}], B = [b_{ij}]$, and denote by $e_1, ..., e_n$ the standard basis for $\mathbb{R}^n$. Then we have, for all $1 \leq i, j \leq n$, that $a_{ij} = e_i^T A e_j = e_i^T B e_j = b_{ij}$, so $A = B$.

7 7.1.4

Normalizing $(1,0)$, for one basis vector we can use $(\sqrt{2}^{-1}, 0)$. One of the conditions the other vector must satisfy is:

$$(\sqrt{2}^{-1}, 0)^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} (x, y) = 0.$$

One such vector is $(x, y) = (\sqrt{5}^{-1}, -2\sqrt{5}^{-1})$, which is actually orthonormal to the first, as you can check. These two vectors, $(\sqrt{2}^{-1}, 0), (\sqrt{5}^{-1}, -2\sqrt{5}^{-1})$ are clearly linearly independent, and so we have an orthonormal basis.

8 7.1.6

If the field F does not have characteristic 2, then $2^{-1} \in F$. We have

$$2^{-1}(q(v+w) - q(v) - q(w)) = 2^{-1}(< v+w, v+w > - < v, v > - < w, w >) =$$

$$2^{-1}(< v, v > +2 < v, w > + < w, w > - < v, v > - < w, w >) =$$

$$< v, w > .$$

9 7.2.4

The given vector along with the two vectors $(1,0,0), (0,1,0)$ clearly form a basis for $\mathbb{R}^3$. We apply the Gram-Schmidt process to this basis. Let $u_1 = \frac{1}{\sqrt{3}}(1,1,1)$.

$$w_2 = (1,0,0) - < \frac{1}{\sqrt{3}}(1,1,1), (1,0,0) \frac{1}{\sqrt{3}}(1,1,1) =$$

$$(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}).$$

Normalizing, we have the second vector of our orthonormal basis as $\sqrt{\frac{1}{6}}(2,-1,-1) = u_2$. For the third vector, we compute:

$$w_3 = (0,1,0) - < \frac{1}{\sqrt{3}}(1,1,1), (0,1,0) > \frac{1}{\sqrt{3}}(1,1,1)$$

$$- < \sqrt{\frac{1}{6}}(2,-1,-1), (0,1,0) > \sqrt{\frac{1}{6}}(2,-1,-1) =$$

$$(0, \frac{1}{2}, -\frac{1}{2}).$$

Normalizing, we have $u_3 = (0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. Thus u_1, u_2, u_3 form the required orthonormal basis.

10 7.2.6

Because P is invertible, we have the equation $(P^{-1})^T A = A'P$. This equation implies that if $x \in kerA$ then $Px \in kerA'$. Note P is invertible, so this sets up a bijective correspondence between the kernel of A and the kernel of A', which implies their ranks are equal.

11 7.2.7

Disprove. Let $A = I$ be the matrix of the standard dot product in $\mathbb{R}^n$. It clearly has n ones for eigenvalues. The matrix of A with respect to an arbitrary basis is $P^T AP$ for some invertible P. One such change of basis matrix is $P = 2I$. We have $P^T AP = 4I$, which clearly has n fours for eigenvalues. Therefore, the eigenvalues of A are not independent of basis.