Bartek Czech
Problem Set 8 Solutions
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1 2 / 5.8.

Suppose the degrees of p_n are bounded. I show that f must then be a polynomial.

If the degrees of the p_n are bounded, then at least one, call it d, appears infinitely many times. Define the subsequence p_{n_k} as containing those polynomials amongst the p_n, whose degree is d.

Because $p_n \to f$ uniformly, $p_{n_k} \to f$ uniformly.

Select $d+1$ distinct points in $[0,1]$, say $x_0, ...x_d$. Then, defining the π's as in the book,

$$p_{n_k}(x) = \sum_{i=0}^{d} \frac{\pi_i(x)}{\pi_i(x_i)} p_{n_k}(x_i)$$

Since uniform convergence implies pointwise convergence, $p_{n_k}(x_i) \to f(x_i)$. Thus:

$$p_{n_k}(x) = \sum_{i=0}^{d} \frac{\pi_i(x)}{\pi_i(x_i)} p_{n_k}(x_i) \to \sum_{i=0}^{d} \frac{\pi_i(x)}{\pi_i(x_i)} f(x_i)$$

Thus, by the uniqueness of limits, $f = \sum_{i=0}^{d} \frac{\pi_i(x)}{\pi_i(x_i)} f(x_i)$ is a polynomial.

2 3 / 5.8.

It suffices to prove that $\{non-polynomials\}$ is not closed, i.e. it does not contain all of its accumulation points. In other words, it suffices to show that there is a sequence of non-polynomials $f_n \to p$ a polynomial (uniformly).

$f_n(x) = 1 + \frac{\sin x}{n}$ satisfy this condition. ($p = 1$, a polynomial.)

A subset of a metric space can easily be both dense and open. $R - Z$ is an example of an open dense subset of the metric space R. Also, any metric space is dense and open in itself.

3 4 / 5.8.

Because $[0,1] \times [0,1]$ is compact (closed and bounded in R^2), by 5.8.2. it is enough to show that $P = \{x, y-polynomials\}$ is an algebra containing constants and separating points.

If $p_1(x,y)$ and $p_2(x,y)$ are polynomials in x and y, then so are $p_1(x,y)p_2(x,y)$, $p_1(x,y) + p_2(x,y)$, and $\alpha p_1(x,y)$ for all $\alpha \in R$. Thus P forms an algebra.

Also, $(p_c(x,y) = 1) \in P$.

Furthermore, at least one of ($p_x(x,y) = x$) and ($p_y(x,y) = y$) separates any pair of distinct (x_1, y_1) and (x_2, y_2).

Thus the conditions of Stone-Weierstrass are satisfied and the proof is complete.

4 50 / Chapter 5.

I first show that $\exists N | n, m > N \Rightarrow p_n - p_m$ is a constant polynomial.

Suppose not, i.e. $\forall N, \exists n, m > N | (p_n - p_m)$ is not constant. But then $\|p_n - p_m\| = \infty$, because every non-constant polynomial diverges as $x \to \infty$. So $\forall N, \exists n, m > N | \|p_n - p_m\| = \infty$. But then it is impossible that $p_n \to f$. Thus if $p_n \to f$, then past a certain N, the p_n differ only in constant terms.

Chop off the beginning N terms of the sequence p_n (skipping the first finitely many terms does not influence the limit), i.e. define $q_n = p_{N+n}$. $q_n \to f$ uniformly.

Now $q_n = a_r x^r + ...a_1 x + b_n$. Then $q_n \to a_r x^r + ...a_1 x + \lim_{n\to\infty} b_n$, which is a polynomial (we know $\lim_{n\to\infty} b_n$ exists, because we're given that $q_n \to f$ uniformly.) By uniqueness of limits,

$$f = a_r x^r + ...a_1 x + \lim_{n\to\infty} b_n, \text{ a polynomial. QED.}$$

5 2 / 6.2.

$f = \exp(x^2 + y^2 + z^2)$. $\mathrm{D}f = 2\exp(x^2 + y^2 + z^2)\ (x\ y\ \epsilon z)$. $\nabla f = 2\exp(x^2 + y^2 + z^2)(x, y, z)$.

6 3 / 6.2.

It is easy to see that the derivative is additive: suppose α, β are two functions differentiable at x_0. Then, given $\epsilon \geq 0$, $\exists \delta > 0|$ if $\|x - x_0\| < \delta_1$, then:

$$\|\alpha(x) - \alpha(x_0) - \mathrm{D}\alpha(x_0)(x - x_0)\| \leq \frac{\epsilon}{2}\|x - x_0\|$$

and similarly with δ_2 for β. Then for $\|x - x_0\| < \min\{\delta_1, \delta_2\}$:

$$\begin{aligned} 0 &\leq \|(\alpha+\beta)(x) - (\alpha+\beta)(x_0) - \mathrm{D}\alpha(x_0)(x - x_0) - \mathrm{D}\beta(x_0)(x - x_0)\| \leq \\ &\leq \|\alpha(x) - \alpha(x_0) - \mathrm{D}\alpha(x_0)(x - x_0)\| + \|\beta(x) - \beta(x_0) - \mathrm{D}\beta(x_0)(x - x_0)\| \leq \epsilon\|x - x_0\| \end{aligned}$$

So indeed, $\mathrm{D}(\alpha + \beta)(x_0) = \mathrm{D}\alpha(x_0) + \mathrm{D}\beta(x_0)$.

By Example 6.2.4., $\mathrm{D}L(x) = L$ for all x. Thus, given the definition $f(x) = L(x) + g(x)$ and the additivity of the derivative shown above, to show that $\mathrm{D}f(0) = L$ it suffices to show that $\mathrm{D}g(0) = 0$. To see that, given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2M}$. Then for $\|x\| < \delta$

$$\|g(x) - g(0) - 0\| = \|g(x)\| \leq M\|x\|^2 \leq \epsilon\|x\| \Leftrightarrow \epsilon \geq M\|x\|$$

is satisfied and so $\mathrm{D}g(0) = 0$ according to the definition of the derivative given at the top of p.329 in the book. (In here, I used $0 \leq \|g(0)\| \leq M \times 0^2 = 0$, so $g(0) = 0$.) The proof is complete.

7 1 / 6.3.

$f(x)$ satisfies the conditions of the previous exercise with $L = 0$ and $M = 1$. Thus $\mathrm{D}f(0) = 0$. The continuity of f at 0 follows from its differentiability.

8 3 / 6.3.

Consider $f(x) = -|x|$. This attains a maximum at $x = 0$, but is not differentiable at 0.

9 4 / 6.4.

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x \sin(\frac{1}{x}) = 0 = f(0)$, so f is continuous at zero.

Because f is even, $\mathrm{D}f(0) = 0$ is the only possible scenario. However,

$$\lim_{x\to 0} \frac{\|f(x) - f(0)\|}{\|x\|} = \lim_{x\to 0} \sin\frac{1}{x}$$

does not exist, so the definition 6.1.1. is not satisfied and f is not differentiable at 0.