Problem Set 7 Solution Set

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1. *Exercise 5.4.1.* Verify that $\int_0^x \sin t\, dt = 1 - \cos x$, using the series expansion.

   Solution. We know that the series

$$\sin t = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1}$$

   converges uniformly on any bounded subset of $\mathbb{R}$. If we choose b such that $|x| < b$, then we have uniform convergence on $[-b, b]$, so we can integrate term-by-term between 0 and x:

$$\begin{aligned} \int_0^x \sin t\, dt &= \int_0^x \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \right) dt = \sum_{k=0}^{\infty} \left(\int_0^x \frac{(-1)^k}{(2k+1)!} t^{2k+1} dt \right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2(k+1))!} x^{2(k+1)} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m)!} t^{2m} \\ &= -(\cos x - 1) = 1 - \cos x \end{aligned}$$

   □

2. *Exercise 5.4.8.* The "error function" is defined by $\operatorname{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt$.

   (a) Show that $\operatorname{erf}(x)$ can be represented by a power series $\sum_{k=0}^{\infty} a_k x^k$ valid for all x, and compute a_0, a_1, a_2, a_3, a_4, and a_5.

   Solution. Since $e^t = \sum_0^{\infty} t^n/n!$, we have

$$e^{-t^2/2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{2^n \cdot n!}$$

   Consider the interval $[-a, a]$ and set $M_n = \sup_{t\in[-a,a]} \left| \frac{t^{2n}}{2^n \cdot n!} \right| = \frac{a^{2n}}{2^n \cdot n!}$. The ratio test shows that $\sum M_n$ converges, since

$$\frac{M_{n+1}}{M_n} = \left| \frac{a^{2(n+1)}}{2^{(n+1)} \cdot (n+1)!} \right| \cdot \left| \frac{2^n \cdot n!}{a^{2n}} \right| = \frac{a^2}{2(n+1)}$$

converges to zero, which is less than 1. Hence, by the Weierstrass M test, we have uniform convergence on $[-a, a]$. Since a was arbitrary, we have uniform convergence on all of $\mathbb{R}$. Thus, on the finite interval $[0, x]$,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}dt &= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\left(\int_0^x (-1)^n \frac{t^{2n}}{2^n\cdot n!}dt\right) \\ &= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)\cdot 2^n\cdot n!}\end{aligned}$$

We compute $a_0 = 0$, $a_1 = \frac{1}{\sqrt{2\pi}}$, $a_2 = 0$, $a_3 = -\frac{1}{6\sqrt{2\pi}}$, $a_4 = 0$, $a_5 = \frac{1}{40\sqrt{2\pi}}$. □

(b) Estimate the value of $\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-x^2/2}dx$.

Solution. We compute

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-x^2/2}dx &= \frac{1}{\sqrt{2\pi}}\left(\int_0^1 e^{-x^2/2}dx - \int_0^{-1} e^{-x^2/2}dx\right) \\ &= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n(1)^{2n+1} - (-1)^n(-1)^{2n+1}}{(2n+1)\cdot 2^n\cdot n!} \\ &= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n - (-1)^{3n+1}}{(2n+1)\cdot 2^n\cdot n!} \\ &\approx \frac{1}{\sqrt{2\pi}}\left(2 - \frac{2}{6} + \frac{2}{40} - \frac{2}{336}\right) \approx 0.68\end{aligned}$$

Indeed, it is the case that roughly 68% of a normally distributed population lies within one standard deviation of the mean. □

3. *Exercise 5.5.4.* Let $f_n(x) = \frac{1}{n}\frac{nx}{1+nx}$, for $0 \le x \le 1$. Show that $f_n \to 0$ in $\mathcal{C}([0,1],\mathbb{R})$.

Solution. $f_n \to 0 \Leftrightarrow \|f_n - 0\|_{[0,1]} \to 0$ as $n \to \infty \Leftrightarrow \sup_{[0,1]}|f_n(x) - 0| \to 0$ as $n \to \infty$. Therefore, we examine

$$\sup_{[0,1]}\left|\frac{1}{n}\frac{nx}{1+nx}\right|$$

Note that f_n is an increasing function on $[0, 1]$ since

$$\begin{aligned}f_n'(x) &= \frac{1}{n}\cdot\frac{n(1+nx) - n^2x}{(1+nx)^2} \\ &= \frac{1}{n}\cdot\frac{1}{(1+nx)^2} > 0\end{aligned}$$

Therefore

$$\sup_{[0,1]}\left|\frac{1}{n}\frac{nx}{1+nx}\right| = \frac{1}{1+n}$$

If we choose $N = (1-\epsilon)/\epsilon$, then for $n > N$, $\dfrac{1}{1+n} < \dfrac{1}{1+N} = \epsilon$, so $\dfrac{1}{1+n} \to 0$ as $n \to \infty$. Thus $f_n \to 0$. $\square$

4. *Exercise 5.7.3.* For what intervals $[0, r]$, $r \leq 1$, is $f : [0, r] \to [0, r]$, $x \mapsto x^2$, a contraction?

   Solution. We want to determine r such that there exists a constant k, $0 \leq k < 1$, such that $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in [0, r]$. Without loss of generality, assume that $x > y$. Since

   $$d(f(x), f(y)) = |x^2 - y^2| = |x + y||x - y| = |x + y|d(x, y)$$

   we want $|x + y| \leq k < 1$ for all $x, y \in [0, r]$. We claim that this implies $r < 1/2$. Indeed, if we were to let $r = 1/2$, then we could choose $x = 1/2$ and $y = 1/2 - \epsilon$ where $\epsilon > 0$. As $\epsilon \to 0$, $|x + y| \to 1$, so we would have to choose $k = 1$ so that $|x + y| \leq k$ for all x, y. But this is not allowed, so we must restrict $r < 1/2$. $\square$

5. *Exercise 5.7.5.* Convert $dy/dx = 3xy$, $y(0) = 1$, to an integral equation and set up an iteration scheme to solve it.

   Solution. The equation is equivalent to $f(x) = 1 + \int_0^x 3tf(t)dt$. Define a map $\Phi : \mathcal{C}[-r, r] \to \mathcal{C}[-r, r]$ by $\Phi(f)(x) = 1 + \int_0^x 3tf(t)dt$. We are in search of a point such that $\Phi(f) = f$. We know that such a point exists because Φ is a contraction mapping and must therefore have a unique fixed point. Suppose we start with $f_0(x) = 1$. Then

   $$\begin{aligned} f_1(x) &= \Phi(f_0)(x) = 1 + \int_0^x 3t dt = 1 + \frac{3x^2}{2} \\ f_2(x) &= \Phi(f_1)(x) = 1 + \int_0^x 3t\left(1 + \frac{3t^2}{2}\right) dt = 1 + \frac{3x^2}{2} + \frac{9}{(2)(4)}x^4 \end{aligned}$$

   Continuing with this iterative process, we will arrive at an infinite series that converges to $f(x) = e^{3x^2/2}$. $\square$

6. *End of Chapter 5, Exercise 2.* Determine which of the following sequences converge (pointwise or uniformly) as $k \to \infty$. Check the continuity of the limit in each case.

   (a) $(\sin x)/k$ on $\mathbb{R}$. Since $(\sin x)/k \leq 1/k$ for all $x \in \mathbb{R}$ and since $(1/k) \to 0$ as $k \to \infty$, $(\sin x)/k$ converges pointwise to 0. To test for uniform convergence, we calculate

   $$\sup_{x \in \mathbb{R}} \left| \frac{\sin x}{k} - 0 \right| = \sup_{x \in \mathbb{R}} \left| \frac{\sin x}{k} \right| = \frac{1}{k}$$

   If $N(\epsilon) = 1/\epsilon$, then for $k > N(\epsilon)$, $1/k < 1/N = \epsilon$. So we have uniform convergence. The limit function $f(x) = 0$ is continuous.

   (b) $1/(kx + 1)$ on $(0, 1)$. We claim that this converges pointwise to the continuous limit function $f(x) = 0$ for $x \in (0, 1)$. Fix $x = \xi$. Given $\epsilon > 0$,

   $$\left| \frac{1}{k\xi + 1} - 0 \right| = \frac{1}{k\xi + 1} < \epsilon$$

for $k > (1-\epsilon)/\xi\epsilon$. Thus pointwise convergence is established. Convergence is not uniform though. If we choose $\epsilon = 1/2$, then we can find $x = \eta \neq 0$ at which

$$\frac{1}{k\eta + 1} > \frac{1}{2}$$

This is in fact true for all points $x = \eta$ where $0 < \eta < 1/k$. Therefore it is impossible to choose n large enough so that $|f_n(x) - f(x)| < \epsilon = 1/2$ for all $x \in (0,1)$.

(c) $x/(kx+1)$ on $(0,1)$. To show pointwise convergence, we argue that $f_k(x) = x/(kx+1)$ is increasing on $(0,1)$ since

$$f_k^{'}(x) = \frac{(1+kx) - kx}{(1+kx)^2} = \frac{1}{(1+kx)^2} > 0$$

Therefore $x/(kx+1) < 1/(k+1)$, which converges to 0 as $k \to \infty$. To establish uniform convergence, we observe that

$$\sup_{x \in (0,1)} \left| \frac{x}{kx+1} - 0 \right| = \frac{1}{k+1}$$

So if we let $N(\epsilon) = (1-\epsilon)/\epsilon$, then for $k > N(\epsilon)$, $1/(k+1) < 1/(1+N(\epsilon)) = \epsilon$. Thus the sequence of functions in question converges uniformly to 0. The limit function $f(x) = 0$ is continuous.

(d) $x/(1+kx^2)$ on $\mathbb{R}$. This clearly converges pointwise to $f(x) = 0$, using a similar argument to part (b). To establish uniform convergence, we argue that $f_k = x/(1+kx^2)$ attains a maximum/minimum when

$$f_k' = \frac{\left(1+kx^2\right) - x\left(2kx\right)\right)}{\left(1+kx^2\right)^2} = 0$$

which implies $1 - kx^2 = 0 \Rightarrow x = \pm 1/\sqrt{k}$. Then

$$\sup_{x \in \mathbb{R}} \left| \frac{x}{1+kx^2} \right| = \frac{1/\sqrt{k}}{1+k(1/k)} = \frac{1}{2\sqrt{k}}$$

If we set $N(\epsilon) = 1/4\epsilon^2$, then $\dfrac{1}{2\sqrt{k}} < \dfrac{1}{2\sqrt{N(\epsilon)}} = \epsilon$. Thus, we have established uniform convergence to the continuous function 0.

(e) $(1, (\cos x)/k^2)$, a sequence of functions from $\mathbb{R}$ to $\mathbb{R}^2$. We claim that this sequence of functions converges uniformly (and thus pointwise) to $(1,0)$. We have

$$\begin{aligned} \sup_{x \in \mathbb{R}} \left\| \left(1, \frac{\cos x}{k^2}\right) - (1,0) \right\| &\leq \left\| \left(1, \frac{1}{k^2}\right) - (1,0) \right\| \\ &= 1/k^2 \end{aligned}$$

If we choose $N(\epsilon) = 1/\sqrt{\epsilon}$, then for $k > N(\epsilon)$, $1/k^2 < 1/N^2 = \epsilon$. Thus the convergence is uniform. The limit function $f(x) = (1,0)$ is a continuous mapping.

7. *End of Chapter 5, Exercise 4.* Let $f_n : [1, 2] \to R$ be defined by $f_n(x) = x/(1 + x)^n$.

   (a) Prove that $\sum_{n=1}^{\infty} f_n(x)$ is convergent for $x \in [1, 2]$.

   Solution. To show convergence, we use the ratio test.

$$\begin{aligned} \lim_{n\to\infty} \sup_{x\in[1,2]} \left|\frac{f_{n+1}}{f_n}\right| &= \lim_{n\to\infty} \sup_{x\in[1,2]} \left|\frac{x}{(1+x)^{n+1}}\right| \left|\frac{(1+x)^n}{x}\right| \\ &= \lim_{n\to\infty} \sup_{x\in[1,2]} \frac{1}{1+x} = \frac{1}{2} < 1 \end{aligned}$$

□

   (b) Is it uniformly convergent?

   Solution. Take $M_n = (2/3)^n$. Note that

$$\begin{aligned} \sup_{x\in[1,2]} |f_n(x)| &= \sup_{x\in[1,2]} \left|\frac{x}{(1+x)^n}\right| \\ &\leq \sup_{x\in[1,2]} \frac{x^n}{(1+x)^n} = \sup_{x\in[1,2]} \left(\frac{x}{1+x}\right)^n \\ &= \left(\frac{2}{3}\right)^n = M_n \end{aligned}$$

   Furthermore, $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series, since $r < 1$. Thus, by the Weierstrass M test, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[1, 2]$. □

   (c) Is $\int_1^2 (\sum_1^{\infty} f_n(x))\, dx = \sum_1^{\infty} \int_1^2 f_n(x)dx$? Since $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[1, 2]$, and since $x/(! + x)^n$ is integrable (it is continuous), it is legitimate to interchange integration and summation.

8. *End of Chapter 5, Exercise 9.* Suppose that the functions g_k are continuous and $\sum_{k=1}^{\infty} g_k$ converges uniformly on $A \subset \mathbb{R}^n$. If $x_k \to x_0$ in A, prove that $\sum_{n=1}^{\infty} g_n(x_k) \to \sum_{n=1}^{\infty} g_n(x_0)$ as $k \to \infty$.

   Solution. Let $g(x) = \sum_{k=1}^{\infty} g_k$. Since the functions g_k are continuous and since $\sum_{k=1}^{\infty} g_k \to g$ uniformly on A, it follows from Corollary 5.1.5 that $g(x)$ is continuous. Since g is continuous, for each convergent sequence $x_k \to x_0$ in A, we have $g(x_k) \to g(x_0)$. □

9. *End of Chapter 5, Exercise 11.*

   (a) Must a contraction on any metric space have a fixed point? Discuss.

   Solution. No. If the metric space is not complete, then the fixed point need not be contained in the metric space. □

(b) Let $f : X \to X$, where X is a complete metric space (such as $\mathbb{R}$), satisfy $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$. Must f have a fixed point? What if X is compact?

Solution. Consider $f(x) = x + 1/x$ on the interval $A = [2, \infty)$. Without loss of generality, we can pick $x, y \in A$ such that $x > y$. Then $1/x < 1/y$. Thus

$$d(f(x), f(y)) = \left| x + \frac{1}{x} - y - \frac{1}{y} \right| = \left| (x - y) + (\frac{1}{x} - \frac{1}{y}) \right| < |x - y| = d(x, y)$$

But we claim that f does not have a fixed point $a \in A$. Indeed, if $f(a) = a$, then $a = a + 1/a \Rightarrow 0 = 1/a$, which is not possible.

If, however, the space X is compact, then we claim that $f : X \to X$ must have a fixed point. Let $g(x) = d(x, f(x))$. Then $g : X \to R$ is a continuous function on a compact space, so it achieves its minimum. That is, there is an $x_0 \in X$ such that $g(x_0) \leq g(x)$ for all $x \in X$. We now claim that $g(x_0) = 0$. If $g(x_0) \neq 0$, then

$$0 \leq g(f(x_0)) = d\left(f\left(x_0\right), f\left(f\left(x_0\right)\right)\right) < d(x_0, f(x_0)) = g(x_0)$$

But this contradicts the fact that $g(x_0)$ is a minimum. Therefore it must be the case that $g(x_0) = 0$, which implies $f(x_0) = x_0$. So f has a fixed point. ☐