Bartek Czech
Problem Set 6 Solutions
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1 38 / Chapter 4

We're given that on $]a, b[$ f has a continuous second derivative, and $f" > 0$.

We want to show that for all x, y in $]a, b[$,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

To interpret this geometrically, we want to show that if $f" > 0$, then the straight line connecting any two points of the graph of f goes above the graph itself, between these two points.

(Parabolas or exponential functions are examples.)

Since $f'' > 0$, we know f' is increasing on $]a, b[$.

Select y,x in $]a, b[$. If the inequality works for $y \leq x$, then by plugging in $s = 1 - t$ in the place of t, we see that it will also work for $y \geq x$. So we may assume without loss of generality that $y \leq x$.

By Mean Value Theorem, there exists a c in [y,x] such that $f'(c) = \frac{f(x)-f(y)}{x-y}$.

Define $z := (x - y)t + y$, i.e. $t = \frac{z-y}{x-y}$. Then the inequality to be proved reads:

$$f(z) \leq f(y) + f'(c)(z - y) = f(x) + f'(c)(z - x)$$

with z in [y,x].

Suppose $z \in [y, c]$. Then by Mean Value Theorem, there exists a $k \in [y, c]$ such that $f(z) = f(y) + f'(k)(z - y)$. But then we're done, because we have:

$$f(z) = f(y) + f'(k)(z - y) \leq f(y) + f'(c)(z - y)$$

as claimed, since $f'(k) \leq f'(c)$ (because $k \leq c$ and f' is increasing,) and $z - y$ is non-negative.

Now suppose $z \in [c, x]$. Then again, by Mean Value Theorem, there exists a $k \in [c, x]$ such that $f(z) = f(x) + f'(k)(z - x)$. But then again we're done, now with:

$$f(z) = f(x) + f'(k)(z - x) \leq f(x) + f'(c)(z - x)$$

as claimed, because $f'(k) \geq f'(c)$ (because $k \geq c$ and f' is increasing,) and $z - x$ is non-positive.

So the claim holds on all of $[y, x]$ and the proof is complete.

2 45 / Chapter 4

Call $\min_{x\in[a,b]} g(x) = m$ and $\max_{x\in[a,b]} g(x) = M$. Because $[a,b]$ is compact and g continuous, $\exists t, T \in [a,b]$ such that $g(t) = m$ and $g(T) = M$.

Define $h(s) = g(s)\int_a^b f(x)dx$. h is clearly continuous on $[a,b]$.

$f(x)m \leq f(x)g(x) \leq f(x)M$ because $f \geq 0$, so by Proposition 4.8.5.iii. (book)

$$h(t) = m\int_a^b f(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b f(x)dx = h(T)$$

Then by the Intermediate Value Theorem, $\exists x_0 \in [a,b]$ such that $\int_a^b f(x)g(x)dx = h(x_0)$, i.e. $\int_a^b f(x)g(x)dx = g(x_0)\int_a^b f(x)dx$.

I now show that a choice $x_0 \in (a,b)$ is always possible.

If $g(x_0) \neq m$ and $g(x_0) \neq M$, then we can apply the Intermediate Value Theorem to g confined to the interval (t,T) or (T,t):

Since $g(t) < g(x_0) < g(T)$, $\exists x_0' \in (t,T) \subset (a,b)$ (or $x_0' \in (T,t) \subset (a,b)$) with $g(x_0') = g(x_0)$. Thus

$$\int_a^b f(x)g(x)dx = g(x_0')\int_a^b f(x)dx$$

and we're done. So suppose $g(x_0) = m$ or $g(x_0) = M$.

If $g(x_0) = m$, then

$$m\int_a^b f(x)dx = \int_a^b f(x)g(x)dx \Longrightarrow \int_a^b f(x)[g(x)-m]dx = 0.$$

But because $f(x) \geq 0$ and $g(x) - m \geq 0$ by definition of m, this implies that $f(x)[g(x)-m] = 0$ on a dense subset of $[a,b]$. Now if $f(x) = 0$ on a dense subset of $[a,b]$, then $\int_a^b f(x)g(x)dx = 0$ and any $x_0 \in (a,b)$ works. So suppose that the set $\{x|f(x) = 0\}$ is not dense in $[a,b]$. Then the set $\{x|g(x) = m\}$ must be infinite, and so in particular it must contain a point x_0' other than a, b, i.e. contained in (a,b). So again:

$$\int_a^b f(x)g(x)dx = m\int_a^b f(x)dx = g(x_0')\int_a^b f(x)dx$$

And identical argument works in the other case. Suppose $g(x_0) = M$. Then:

$$M\int_a^b f(x)dx = \int_a^b f(x)g(x)dx \Longrightarrow \int_a^b f(x)[M - g(x)]dx = 0.$$

But because $f(x) \geq 0$ and $M - g(x) \geq 0$ by definition of M, this implies that $f(x)[M - g(x)] = 0$ on a dense subset of $[a, b]$. Now if $f(x) = 0$ on a dense subset of $[a, b]$, then $\int_a^b f(x)g(x)dx = 0$ and any $x_0 \in (a, b)$ works. So suppose that the set $\{x|f(x) = 0\}$ is not dense in $[a, b]$. Then the set $\{x|g(x) = M\}$ must be infinite, and so in particular it must contain a point x_0' in (a, b). Thus:

$$\int_a^b f(x)g(x)dx = M\int_a^b f(x)dx = g(x_0')\int_a^b f(x)dx$$

Now we see that a choice $x_0' \in (a, b)$ is always possible and the proof is complete.

A lot of you only showed that a choice $x_0 \in [a, b]$ is possible, or failed to take into account that we're given $f \geq 0$, not $f > 0$. Given how much effort it takes to show that not necessarily $x_0 = a$ or b, I ascribed as many as 4/10 points to the latter parts of the problem.

3 1 / Section 5.1.

For all $0 \leq x \leq 1$, $f_n(x) \to x^2$. So to show that the f_n converge uniformly, it's enough to show that given $\epsilon > 0$, there exists an N such that

$$n > N \Longrightarrow |f_n(x) - x^2| < \epsilon$$

for all $0 \leq x \leq 1$. Set $N = \max\{1, \frac{3}{\epsilon}\}$. Then

$$n > N \Longrightarrow |f_n(x) - x^2| = |-\frac{2x}{n} + \frac{1}{n^2}| \leq \frac{2x}{n} + \frac{1}{n^2} \leq \frac{2}{n} + \frac{1}{n^2} \leq \frac{2}{n} + \frac{1}{n} \leq \frac{3}{n} < \frac{3}{N} \leq \epsilon$$

so the convergence is uniform.

4 4 / Section 5.1.

For all $0 \leq x \leq 0.999$, $f_n(x) \to 0$. So to show that the f_n converge uniformly, it's enough to show that given $\epsilon > 0$, there exists an N such that

$$n > N \Longrightarrow |f_n(x)| = x^n < \epsilon$$

for all $0 \leq x \leq 0.999$. Set $N = \log_{0.999} \epsilon$. Then:

$$n > N \Longrightarrow x^n \leq 0.999^n < 0.999^N = 0.999^{\log_{0.999} \epsilon} = \epsilon$$

and again, the convergence is indeed uniform. Note the difference from Example 5.1.8./241 from the book.

5 4 / Section 5.2.

Set $g_n(x) = \frac{1}{n^2+x^2}$. Then $|g_n(x)| \leq M_n = \frac{1}{n^2}$ for all $x \in R$. Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series), by the Weierstrass M-test (5.2.2./245), $\sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$ converges uniformly on all of R.

5.1 5 / Section 5.2.

Set $g_n(x) = a_n \sin nx$. Since $|\sin nx| \leq 1$, $|g_n(x)| \leq |a_n|$. Since $\sum_{n=1}^{\infty} |a_n|$ converges ($\sum_{n=1}^{\infty} a_n$ converges absolutely), $\sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} a_n \sin nx$ converges uniformly by the Weierstrass M-test, as desired.