Problem Set 5 Solution Set

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(adapted from Nitin Saksena's LaTeXfile)

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1. *End of Chapter 4, Exercise 6b.* Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. Prove that f is continuous if and only if the graph of f is a closed subset of $\mathbb{R}^2$. What if f is unbounded?

   Solution. ($\Longrightarrow$) Assume f is continuous. Let $G_f = \{(x, f(x)) \mid x \in \mathbb{R}\}$ denote the graph of f. Since f is continuous, for each convergent subsequence $x_k \to x_0$, we have $f(x_k) \to f(x)$. Thus every sequence of the form $(x_n, f(x_n)) \in G_f$ converges to $(x, f(x)) \in G_f$. But this means that G_f is closed.

   ($\Longleftarrow$) Assume that G_f is closed. Since f is bounded, say $|f(x)| \leq M$, every sequence $(x_n, f(x_n))$ in G_f such that $x_n \to x$ is contained in $(\{x_n\} \cup \{x\}) \times [-M, M]$. This is the product of two compact spaces and so it is compact in $\mathbb{R}^2$. Hence $(x_n, f(x_n))$ has a convergent subsequence $(x_{n_k}, f(x_{n_k}))$ in this space. Since the limit point lies in G_f, it must be of the form $(x, f(x))$. Hence $f(x_{n_k}) \to f(x)$ and so f is continuous.

   If f is unbounded then G_f could be closed but f could be discontinuous. Consider the function

   $$f(x) = \begin{cases} 1/x & x > 0 \\ 0 & x \leq 0 \end{cases}$$

   The graph of this function is closed (its complement is open–also, it is the union of two graphs that are closed). But the function is discontinuous at zero. $\square$

2. *End of Chapter 4, Exercise 7.* Consider a compact set $B \subset \mathbb{R}^n$ and let $f : B \to \mathbb{R}^m$ be continuous and one-to-one. Then prove that $f^{-1} : f(B) \to B$ is continuous. Show by example that this may fail if B is connected but not compact.

   Solution. Consider the closed set $C \subset B \subset \mathbb{R}^n$. Since B is closed and bounded and C is a closed subset of B, C is also bounded and therefore compact by the Heine–Borel theorem. Since f is continuous, it follows that $f(C)$ is compact and thus closed. We then claim that $(f^{-1})^{-1}(C) = f(C)$. Indeed, since f is one-to-one, it maps B bijectively onto its image $f(B)$. One can then easily show inclusions both ways.

   Hence given a closed set $C \subset B$, $(f^{-1})^{-1}(C)$ is closed, so f^{-1} is continuous.

   To show that this may fail if B is connected but not compact, consider $f : [0, 2\pi) \to \mathbb{R}^2$ given by $f(t) = (\sin t, \cos t)$. Observe that $f([0, 2\pi))$ equals the unit circle $S \subset \mathbb{R}^2$. (Also f is one-to-one and continuous.) But the preimage of f^{-1}, which equals f, maps an open set to a closed set so f^{-1} cannot be continuous. $\square$

3. *End of Chapter 4, Exercise 8.* Define maps $s : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $m : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ as addition and scalar multiplication maps defined by $s(x, y) = x + y$ and $m(\lambda, x) = \lambda x$. Show that these mappings are continuous.

   Solution. Consider the continuous functions $s_1(x, y) = x$ and $s_2(x, y) = y$. Then $s = s_1 + s_2$ and by Corollary 4.3.3, s is continuous. Now consider the continuous functions $m_1(\lambda, x) = \lambda$ and $m_2(\lambda, x) = x$. Then $m = m_1 m_2$ and by Corollary 4.3.3, m is continuous. $\square$

4. *End of Chapter 4, Exercise 18.* Let $A \subset M$ be connected and let $f : A \to \mathbb{R}$ be continuous with $f(x) \neq 0$ for all $x \in A$. Show that $f(x) > 0$ for all $x \in A$ or else $f(x) < 0$ for all $x \in A$.

   Solution I. Suppose there exist points a, b of A for which $f(a) < 0$ and $f(b) > 0$. Then by the Intermediate Value Theorem there exists a point $c \in (a, b)$ such that $f(c) = 0$, contrary to the hypothesis on f. Thus either $f(x) > 0$ for all $x \in A$ or $f(x) < 0$ for all $x \in A$. $\square$

   Solution II. Suppose there exist points a, b of A for which $f(a) < 0$ and $f(b) > 0$. Define the sets

$$U = \{x \in M \mid f(x) < 0\}$$
$$V = \{x \in M \mid f(x) > 0\}.$$

   We are assuming U and V are non-empty. Note that both U and V are open, since, for example, $U = f^{-1}((-\infty, 0))$, i.e., it is the pre-image of an open set under a continuous mapping. Then U and V form a separation of A, contrary to the hypothesis that A is connected. Hence one of the sets U or V must be empty in fact. $\square$

5. *End of Chapter 4, Exercise 29.* Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leq |x - y|^2$. Prove that f is a constant.

   Solution. $|f(x) - f(y)| \leq |x - y|^2 \Rightarrow \dfrac{|f(x) - f(y)|}{|x - y|} \leq |x - y|$ (since $|x - y|$ is non-negative) $\Rightarrow \left|\dfrac{f(y) - f(x)}{y - x}\right| \leq |y - x|$. It follows that $\lim_{y \to x} \left|\dfrac{f(y) - f(x)}{y - x}\right| \leq \lim_{y \to x} |y - x| \Rightarrow |f'(x)| \leq 0 \Rightarrow f'(x) = 0 \Rightarrow f$ is constant. $\square$

6. *End of Chapter 4, Exercise 30a.* Let $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$. Prove that f is uniformly continuous.

   Solution I. We'll prove f is uniformly continuous in $[0, 1/4]$ and on $(1/4, +\infty)$. The former follows from the Uniform Continuity theorem since f is continuous and $[0, 1/4]$ is compact. For the latter, notice $f'(x) = 1/2\sqrt{x} < 1$ for $x > 1/4$. Since f has bounded derivative (i.e., it is Lipshitz), it is uniformly continuous in $(1/4, +\infty)$. $\square$

Solution II. Let $\epsilon > 0$ be given and choose $\delta = \epsilon^2$. We want to show that for every pair of points x, y such that $|x - y| < \delta$, $|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = |\sqrt{y} - \sqrt{x}| < \epsilon$. Observe that $|x - y| < \delta \Rightarrow y < x + \delta \Rightarrow \sqrt{y} < \sqrt{x + \delta}$. Thus

$$|f(x) - f(y)| = |\sqrt{y} - \sqrt{x}| < |\sqrt{x+\delta} - \sqrt{x}| < |\sqrt{x} + \sqrt{\delta} - \sqrt{x}| = \sqrt{\delta} = \epsilon$$

as desired. Note that the inequality $\sqrt{x+y} < \sqrt{x} + \sqrt{y}$ is valid for $x, y \in [0, +\infty)$. □

7. *End Chapter 4, Exercise 34.* Assuming that the temperature on the surface of the earth is a continuous function, prove that on any great circle of the earth there are two antipodal points with the same temperature.

   Solution. Pick a great circle on the earth and a point p on that circle. Let $T(\theta)$ be the temperature at θ rads, measured counter-clockwise from p. Consider the function $S(\theta) = T(\theta) - T(\theta + \pi)$. S is continuous since T is. It follows that $S(0) = T(0) - T(\pi)$ and $S(\pi) = T(\pi) - T(2\pi) = T(\pi) - T(0)$, $S(0) = -S(\pi)$. From the Intermediate Value Theorem, there exists $c \in (0, \pi)$ such that $S(c) = 0$, which implies that $T(c) = T(c + \pi)$. Thus the temperature at two antipodal points is the same. □

8. *End of Chapter 4, Exercise 42.* For $x > 0$ define $L(x) = \int_1^x (1/t)dt$. Prove the following using this definition:

   (a) L **is increasing in** x**.** From part (c), we have $L'(x) = 1/x$ which is positive when x is positive. Hence L is increasing.

   (b) $L(xy) = L(x) + L(y)$**.** Fix $x > 0$ and let $f(z) = L(xz)$. Since $z \mapsto xz$ is differentiable at every $z > 0$ with derivative x, we can use part (c) and the chain rule to get

   $$f'(z) = L'(xz) \cdot x = (1/xz) \cdot x = 1/z = L'(z)$$

   Since $(f - L)' = 0$, $f - L$ is constant. In particular,

   $$(f - L)(y) = (f - L)(1) = f(1) - L(1) = L(x) - 0$$

   since $L(1) = 0$ from part (d). Thus $L(x) = f(y) - L(y) = L(xy) - L(y)$ which implies $L(xy) = L(x) + L(y)$ as desired.

   (c) $L'(x) = 1/x$**.** For $x > 0$, $f : [1, x] \to \mathbb{R}$ (and $f : [x, 1] \to \mathbb{R}$ in the case $0 < x < 1$) given by $f(t) = 1/t$ is continuous and thus satisfies the hypothesis of the Fundamental Theorem of Calculus. Using the version derived in class, we have $L'(x) = F'(x) = f(x) = 1/x$.

   (d) $L(1) = 0$**.** By definition $L(1) = \int_1^1 (1/t)dt = 0$.

   (e) **Properties (c) and (d) uniquely determine** L**. What is** L**?** Suppose there are two functions $L(x)$ and $M(x)$ such that $L'(x) = M'(x) = 1/x$ and $L(1) = M(1) = 0$. Then $(L - M)' = 0 \Rightarrow (L - M)$ is a constant. Thus $(L - M)(1) = 0 \Rightarrow (L - M)(x) = 0$ from which it follows that $L(x) = M(x)$. Therefore properties (c) and (d) uniquely determine L. $L(x) = \log(x)$.

Remark. Many people had trouble with this exercise. In analysis $L(x)$ is the *definition* of the natural logarithm. So we were not allowed to use the 'known' properties of ln to prove the above statements. Also, part (b) required a clever proof since we have not yet proven the change of variables formula for integration.

9. *End of Chapter 4, Exercise 44.* Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable and suppose for every a, b with $0 \le a < b \le 1$ there is a c, $a < c < b$, with $f(c) = 0$. Prove $\int_0^1 f = 0$. Must f be zero? What if f is continuous?

Solution. For any partition $\bigcup_{k=1}^n I_k$ of $[0, 1]$, we know that there exist $x_k \in I_k$ for which $f(x_k) = 0$. Consider

$$s' = \sum_{k=1}^{n} f(x_k)|I_k|$$

Because $f(x_k) = 0$ for all k, we know that $s' = 0$. We also know that $\inf\{f(x) \,|\, x \in I_k\} \le f(x_k) \le \sup\{f(x) \,|\, x \in I_k\}$. Consequently, $\underline{\int_0^1} f \le s' \le \overline{\int_0^1} f$. But the definition of integrability implies $\underline{\int_0^1} f(x)dx = \overline{\int_0^1} f(x)dx$ and thus

$$\int_0^1 f(x)dx = s' = 0.$$

The function f need not be zero. Consider the following function $f : [0, 1] \to \mathbb{R}$:

$$f(x) = \begin{matrix} 1 & x = 1/2 \\ 0 & x \neq 1/2 \end{matrix}$$

This function satisfies the hypotheses of this exercise and integrates to zero (the lower integral is clearly 0 and the upper integral goes to 0 as the partitions of the interval get finer). If f is continuous however, it must be zero. If f were not identically zero, then in order to satisfy the condition that for every a, b with $0 \le a < b \le 1$ there is a c such that $a < c < b$, f would have to have an infinite number of jump discontinuities which would make it discontinuous. $\square$

Remark. Many people tried using the example $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

This function *is not Riemann integrable.* It is only integrable in the sence of Lesbesgue. So when we write $\int f = 0$ we mean the Lebesgue integral of the function is zero. Look at your class notes for details.