Bartek Czech
Problem Set 4 Solutions
March 4, 2002

1 5 / Chapter 3

1.1 (a)

$$\{x \in R^n | \parallel x \parallel < 1\} \subset \cup_n B(0, 1 - \frac{1}{n})$$

Assume there is a finite subcover C, and denote the largest n such that $B(0, 1 - \frac{1}{n})$ is contained in C by N. Then any x with $\parallel x \parallel = 1 - \frac{1}{2N}$ is not contained in the subcover, so none exists.

1.2 (b)

$N = \cup_{n \in Z} B(n, \frac{1}{2})$

2 6 / Chapter 3

By the nested intersection property, we know that $\cap_k F_k \neq \emptyset$. Now suppose there exist two distinct points x, y in $\cap_k F_k$.Set $d := d(x, y)$. Note d > 0. Since $diam(F_k) \to 0, \exists N \mid diam(F_N) < \frac{d}{2}$. But then x and y cannot both belong to F_N, because distances of points within it are bounded above by $\frac{d}{2}$, by the definition of $diam(F_N)$. So x and y cannot both belong to F_N, and consequently to $\cap_k F_k$. The proof is complete.

3 14(b) / Chapter 3

N, Z, {0,1} are all discrete.

($\Longrightarrow$) Say A is finite, and has N elements. Given any open cover C of A, for each element x_n of A, choose a U_n in C that contains it. Then $\cup_{n=1}^N U_n$ is a finite subcover of A. So A is compact.

($\Leftarrow$) Say A is infinite and discrete. Discreteness means that for each $x \in A, \exists$ an open B_x that contains no element of A other than x. Then $\cup_{x \in A} B_x$ is an infinite open cover of A, which has no proper subcover: if we remove any B_x, we fail to cover x. The proof is complete.

4 21 / Chapter 3

4.1 (a)

($\Longrightarrow$) Suppose A is not connected, and consider a separation of A, $U \cup V$. Define $X = U \cap A$ and $Y = V \cap A$. Then X, Y are open relative to A by construction.

They're also closed relative to A, because $X = A - Y$, $Y = A - X$. Furthermore, none of them equals A or $\emptyset$, for then U, V would fail to be a separation. So A and $\emptyset$ are not the only sets open and closed relative to A.

($\Leftarrow$) Say there is a non-empty open and closed set relative to A, other than A, and call it X. For X to be open, there must be an open $U \subset M$ with $X = U \cap A$. For X to be closed, there must be a closed F with $X = F \cap A$. Define $V = M - F$. Then V is open, and V contains $A - X$. Then U and V form a separation of A in M. To see this, note that $A \subset X \cup (A - X)$, $X \subset U$, $A - X \subset V$; also $U \cap V \neq \emptyset$, and $U \supset X$, which is non-empty, and $V \supset A - X$, which is non-empty, as X is not all of A.

4.2 (b)

R^n is connected, because it is path connected: for any $x, y \in R^n$, $f(t) = x + t(y - x)$ is a path from x to y. So by 21(a), the only open and closed subsets it may have (relative to itself) are itself and $\emptyset$.

5 24 / Chapter 3

($\Longrightarrow$) Suppose A is not connected, , and consider a separation of A, $B \cup C$. To see that C contains no accumulation point of B (and by symmetry, vice versa), note that for any $x \in C$, $\exists r > 0$ with $B(x, r) \subset C$, so $B(x, r) \cap B = \emptyset$ and x is not an accumulation point of B. All the other requirements of the theorem for B and C are immediate consequences of B and C being a separation of A.

($\Leftarrow$) Method I.

Let B and C be as in the theorem. It suffices to show that B is closed relative to A. For then by symmetry, C is also closed (relative to A), and so $B = A - C$ is open relative to A. Therefore by #21(a) A is disconnected, because B is both open and closed relative to A (as B is not all of A, $C \cap A \neq \emptyset$, and B is not empty).

To see that B is closed relative to A, consider the closure of B relative to A. Then B is not closed relative to A only if it is not equal to its own closure in A, i.e. if there is an accumulation point of B, which is in A but not in B. But since A is the disjoint union of C and B, that accumulation point would have to be in C, which gives rise to a contradiction. So B is relatively closed to A and we're done.

($\Leftarrow$) Method II.

Let B and C be as in the theorem. Since every point x of C is a positive distance from B (otherwise it would be an accumulation point of B), and vice versa, d(x,B)>0 for all x in C (and vice versa.) So for all x in C, $B(x, \frac{1}{2}d(x, B))$ is an open set. Define $U = \cup_{x \in B} B(x, \frac{1}{2}d(x, C))$ and $V = \cup_{x \in C} B(x, \frac{1}{2}d(x, B))$. Then U and V are disjoint by construction, open (unions of open sets), contain A (because they contain B and C, respectively) and have non-zero intersections with A (because B and C do.) So U and V separate A.

6 28 / Chapter 3

To argue the contrapositive, say x in A is not an accumulation point of A, and that x is not all of A.

{x} is clearly closed, relative to A (and in general.) x being not an accumulation point of A means that $\exists r > 0$ with $B(x,r) \cap A = \{x\}$. Thus {x} is open relative in A. Since {x} is both open and closed relative in A, and it is not all of A (nor empty, of course), then by #21(a) A is disconnected.

7 3 / 4.1.

$A = f^{-1}([0,1])$, and since f is continuous and [0,1] is closed, so is A.

8 5 / 4.1.

Define f(x)=0 for all x in R. Then f(R)={0}, which is not open, even though R is open. The constant function is clearly continuous.

9 4(a) / 4.2.

To argue the contrapositive, suppose A is disconnected, and U, V form a separation. Then $U \times R$ and $V \times R$ are a separation of $A \times B$, so it's disconnected.

10 5 / 4.2.

Suppose $A \times B$ is open. Take any a in A. Then for any b in B, $\exists r > 0$ with $B((a,b),r) \subset A \times B$. So $\{(x,b)|d(x,a) < r\} \subset B((a,b),r) \subset A \times B$, but that is the same as saying $\{x|d(x,a) < r\} \subset B(a,r) \subset A$. So A is open.