Problem Set 3 Solution Set

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1. *End of Chapter 2 Exercise 15(b).* Let A be a subset of a metric space M. Show that $\mathrm{bd}(\mathrm{bd}\, A) \subset \mathrm{bd}\, A$.

   Solution. On the one hand $\mathrm{bd}(\mathrm{bd}\, A) = \mathrm{cl}(\mathrm{bd}\, A) - \mathrm{int}(\mathrm{bd}\, A)$. We claim, however, that $\mathrm{bd}\, A$ is closed. Indeed, $\mathrm{bd}\, A = \mathrm{cl}\, A \cap \mathrm{cl}\, m \backslash A$, so it is the intersection of two closed sets and is therefore closed. Hence $\mathrm{bd}(\mathrm{bd}\, A) = \mathrm{bd}\, A - \mathrm{int}(\mathrm{bd}\, A)$ from which it follows that $\mathrm{bd}(\mathrm{bd}\, A) \subset \mathrm{bd}\, A$. □

2. *End of Chapter 2 Exercise 17.* If $\sum x_m$ converges absolutely in $\mathbb{R}^n$, show that $\sum x_m \sin m$ converges.

   Solution. We have
   $$||x_m \sin m|| = |\sin m| \cdot ||x_m|| \leq ||x_m||.$$
   Hence $\sum ||x_m \sin m|| \leq \sum ||x_m||$. Since $\sum x_m$ converges absolutely, by the Comparison Test, $\sum ||x_m \sin m||$ converges. Since $\mathbb{R}^n$ is complete and normed, absolute convergence of $\sum x_m \sin m$ is enough to show its convergence. □

3. *End of Chapter 2 Exercise 20.* For a set A in a metric space M and $x \in M$, let
   $$d(x, A) = \inf\{d(x, y) < \epsilon\}$$

   (a) Show that $D(A, \epsilon)$ is open

   Solution (due to Soojin Yim). We must show that around every $x \in D(A, \epsilon)$ there is an open ball entirely contained in the set. Since $x \in D(A, \epsilon)$, it follows that $d(x, A) < \epsilon$. Define ϵ' as $\epsilon - d(x, A)$.
   We claim $D(x, \epsilon') \subset D(A, \epsilon)$. Indeed, let $y \in D(x, \epsilon')$. Then
   $$d(y, z) \leq d(x, z) + d(x, y) \quad \forall\ z \in A,$$
   and taking infs over A we obtain
   $$d(y, A) \leq d(x, A) + d(x, y) < d(x, A) + \epsilon' = \epsilon.$$
   Hence $y \in D(A, \epsilon)$, and so $D(x, \epsilon') \subset D(A, \epsilon)$. □

   (b) Let $A \subset M$ and $N_\epsilon = \{x \in M \,|\, d(x, A) < \epsilon\}$, where $\epsilon > 0$. Show that N_ϵ is closed and that A is closed if and only if $A = \cap \{N_\epsilon \,|\, \epsilon > 0\}$.

Solution. First we show N_ϵ is closed. Equivalently, $M \setminus N_\epsilon$ is open. Let $x \in M \setminus N_\epsilon$. Then $d(x, A) > \epsilon$, so choose $\epsilon' = d(x, A) - \epsilon$. Then $D(x, \epsilon') \subset M \setminus N_\epsilon$. Indeed, if $y \in D(x, \epsilon')$, then, by use of triangle inequality (taking infs over A),

$$d(y, A) \geq d(x, A) - d(x, y) > \epsilon' + \epsilon - \epsilon' = \epsilon.$$

Hence $y \in M \setminus N_\epsilon$.

Now we'll show A is closed if and only if $A = \cap \{N_\epsilon \mid \epsilon > 0\}$. If $A = \cap \{N_\epsilon \mid \epsilon > 0\}$, since the N_ϵ are closed then A is the arbitrary intersection of closed sets, which is closed.

Now suppose that A is closed. One the one hand, since $A \subset N_\epsilon$, it follows that $A \subset \cap \{N_\epsilon \mid \epsilon > 0\}$. To show the opposite inclusion, suppose $x \notin A$ but $x \in \cap \{N_\epsilon \mid \epsilon > 0\}$. A closed means $d(x, A) > 0$. Let $\epsilon = d(x, A)$. Then $x \notin N_{\epsilon/2}$ and this contradicts the assumption that $x \in\subset \cap \{N_\epsilon \mid \epsilon > 0\}$. □

4. *End of Chapter 2 Exercise 26.* Define he sequence of numbers a_n by

$$a_0 = 1, \quad a_n = 1 + \frac{1}{1 + a_{n-1}}.$$

Show that a_n is a convergent sequence. Find the limit.

Solution. First, note that a_n is bounded below by 0 and above by 2 (a simple induction will prove these claims). Now we consider the subsequences a_{2n} and a_{2n+1}. We'll show the former is monotonically increasing and the latter is monotonically decreasing. Note that

$$a_{n+1} - a_{n-1} = \frac{-(a_n - a_{n-2})}{(1 + a_n)(1 + a_{n-2})} = \frac{(a_{n-1} - a_{n-3})}{\text{something} > 1}$$

This shows $a_{n+1} - a_{n-1}$ has the same sign as $a_{n-1} - a_{n-3}$. Inductively, $a_{n+1} - a_{n-1}$ has the same sign as $a_2 - a_0$ or $a_3 - a_1$. In fact, the second equality above shows $a_{n+1} - a_{n-1}$ and $a_n - a_{n-2}$ have opposite signs. Computing the first few terms, it follows that a_{2n} is monotonically increasing while a_{2n+1} is monotonically decreasing. Thus each of these subsequences converges, and they both obey the recurrence relation

$$a_{n+1} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + a_{n-1}}}$$

Now that we have established that the subsequences converge, it makes sense to take limits of the above equation. Since both subsequences satisfy the same recurrence relation, they converge to the same value (so a_n converges to this value). We easily compute that $\lim a_n = \sqrt{2}$. □

5. *End of Chapter 2 Exercise 52.* Test the following series for convergence.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{e^{-k}}{\sqrt{k+1}}$. Use the ratio test to get $\displaystyle\lim_{k\to\infty} \left| \frac{e^k \sqrt{k+1}}{e^{k+1}\sqrt{k+2}} \right| = \frac{1}{e} < 1$, so the series converges.

(b) $\sum_{k=0}^{\infty} \frac{k}{\sqrt{k^2+1}}$. Since $\frac{k}{k^2+1} > \frac{k}{k^2+k^2}$, it follows that $\sum_{k=0}^{\infty} \frac{k}{\sqrt{k^2+1}} > \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{k}$, which diverges. So by the comparison test, our initial series diverges.

(c) $\sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{n^2-3n+1}$. We have $\frac{\sqrt{n+1}}{n^2-3n+1} < \frac{\sqrt{n}}{n^2-3n+1} \leq \frac{2}{n^{3/2}}$. This last inequality follows because $n^2-6n+2 \geq 0$ for $n \geq 6$. So by the comparison test using a p-series, $\sum_{n=6}^{\infty} \frac{\sqrt{n+1}}{n^2-3n+1} \leq 2\sum_{n=6}^{\infty} \frac{1}{n^{3/2}}$, so our original series converges.

(d) $\sum_{n=1}^{\infty} \frac{\log(k+1)-\log k}{\arctan(2/k)}$. We'll show, by use of L'Hopital's rule that the terms in this series blow up. Hence the series will be divergent. We have

$$\lim_{k\to\infty} \frac{\log(1+1/k)}{\arctan(2/k)} = \lim_{k\to\infty} \frac{\frac{1}{1+1/k}\cdot\frac{1}{k^2}}{\frac{1}{1+4/k^2}\cdot\frac{8}{k^3}} = \lim_{k\to\infty} \frac{8(k^3+4k)}{k^2+k} = \infty.$$

(e) $\sum_{n=1}^{\infty} \sin(n^{-\alpha}),\ \alpha \in \mathbb{R}, \alpha > 0$. Since $\sin(n^{-\alpha}) < n^{-\alpha}$ by comparison,

$$\sum_{n=1}^{\infty} \sin(n^{-\alpha}) < \sum_{n=1}^{\infty} n^{-\alpha}.$$

If $\alpha > 1$, then the latter series converges by virtue of the p-series test, i.e., the series converges for $\alpha > 1$. If, however, $0 < \alpha \leq 1$ then there is an N for which $\sin n^{-\alpha} > n^{-\alpha}/2$, for $n \geq N$. Therefore

$$\frac{1}{2}\sum_{n=N}^{\infty} n^{-\alpha} > \sum_{n=N}^{\infty} \sin(n^{-\alpha}).$$

Thus our series diverges by comparison with the tail of the p-series.

(f) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$. By use of the ratio test, we get convergence, since

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3}{n^3}\cdot\frac{3^n}{3^{n+1}}\right| = \left|\frac{(n+1)^3}{n^3}\cdot\frac{1}{3}\right| \to \frac{1}{3} < 1.$$

6. *Exercise 3.1.4.* Let $x_k \to x$ be a convergent sequence in a metric space and let $A = \{x_1, x_2, \ldots\} \cup \{x\}$.

(a) Show that A is compact.

Solution. By proposition 2.8.7, we know that the sequence x_k converges to x if and only if every subsequence converges to x. Since A contains all subsequences to x_k that converge to $x \in A$, it is sequentially compact. Hence A is compact by the Bolzano-Weierstrass theorem. □

(b) Verify that every open cover of A has a finite subcover.

Solution. I made the mistake of telling people this follwed from the definition of compactness. The question asked you to *verify* that open covers had finite subcovers. Nobody got penalized for following my misleading advice.
Let $\{U_i\}$ be an open cover of A. One U_α must contain x. Then U_α contains x_n for $n \geq N$, where N is large enough. For each x_n with $n < N$, there must me an element U_{i_n} of the cover that contains x_n. Hence

$$\left(\bigcup U_{i_n}\right) \cup U_\alpha$$

is a finite subcover of A. □

7. *Exercise 3.2.5* Let A bean infinite set in $\mathbb{R}$ with a single accumulationpoint in A. Must A be compact?

Solution. No. Consider the set $A = \{1/n \,|\, n \in \mathbb{N}\} \cup \mathbb{Z}$. This set is infinite and has a single accumulation point $0 \in A$. But it is not bounded. So it is not compact by the Heine–Borel theorem. □