Bartek Czech
Problem Set 2 Solutions
February 20, 2002

1 12a / Chapter 1

$$\begin{aligned} \|x+y\|^2+\|x-y\|^2 &= <x+y,x+y> + <x-y,x-y> \\ &= <x,x> + <x,y> + <y,x> + <y,y> \\ + &< x,x> - <x,y> - <y,x> + <y,y> \\ &= 2<x,x> +2<y,y> = 2\|x\|^2+2\|y\|^2 \end{aligned}$$

as desired. The geometric significance of this law is that the sum of squares of all four sides of a parallelogram equals the sum of squares of its two diameters. In R^2, note the connection with the cosine law.

2 30 / Chapter 1

Consider $f(x)=x, g(x)=1-x$. Then $\|f\|_\infty=1, \|g\|_\infty=1, \|f+g\|_\infty=\|1\|_\infty=1, \|f-g\|_\infty=\|2x-1\|_\infty=1$,and

$$1^2+1^2 \neq 2\cdot 1^2+2\cdot 1^2$$

This suffices to show that $\|\cdot\|_\infty$ does not come from an inner product.

3 2 / Chapter 2

3.1 (a)

$int(]1,2[) =]1,2[$
$cl(]1,2[) = [1,2]$
$bd(]1,2[) = \{1,2\}$
as for any interval. Similarly:

3.2 (b)

$int([2,3]) =]2,3[$
$cl([2,3]) = [2,3]$
$bd([2,3]) = \{2,3\}$

3.3 (c)

$int(\cap_{n=1}^{\infty}[-1,\frac{1}{n}[) =]-1,0[$
$cl(\cap_{n=1}^{\infty}[-1,\frac{1}{n}[) = [-1,0]$
$bd(\cap_{n=1}^{\infty}[-1,\frac{1}{n}[) = \{-1,0\}$
since $\cap_{n=1}^{\infty}[-1,\frac{1}{n}[=]-1,0]$.

3.4 (d)

$int(R^n) = R^n$
$cl(R^n) = R^n$
$bd(R^n) = \emptyset$ in R^n.

3.5 (e)

$int(hyperplane) = \emptyset$
because every ball centered in a point belonging to a (hyper-)plane contains elements outside the hyperplane.
$cl(hyperplane) = hyperplane$
$bd(hyperplane) = hyperplane$

3.6 (f)

$int(\{0 < r < 1 | r \in Q\}) = \emptyset$
because for every rational number r one can construct a sequence of non-rationals converging to r.
$cl(\{0 < r < 1 | r \in Q\}) = [0,1]$
$bd(\{0 < r < 1 | r \in Q\}) = [0,1]$

3.7 (g)

$int(\{(x,y) \in R^2 | 0 < x \leq 1\}) = \{(x,y) \in R^2 | 0 < x < 1\}$
$cl(\{(x,y) \in R^2 | 0 < x \leq 1\}) = \{(x,y) \in R^2 | 0 \leq x \leq 1\}$
$bd(\{(x,y) \in R^2 | 0 < x \leq 1\}) = \{(0,y) \in R^2\} \cup \{(1,y) \in R^2\}$

3.8 (h)

$int(\{x \in R^n | \|x\| = 1\}) = \emptyset$
because any point on the boundary of the 1-ball is an accumulation point of the complement of the boundary. Think of the sequence $(1-\frac{1}{n})x \longrightarrow x$ for $\|x\| = 1$.
$cl(\{x \in R^n | \|x\| = 1\}) = \{x \in R^n | \|x\| = 1\}$
$bd(\{x \in R^n | \|x\| = 1\}) = \{x \in R^n | \|x\| = 1\}$

4 8 / Chapter 2

Let $S \subset R$ be non-empty and bounded below. We know $\inf(S)$ exists. The condition for the inf is that for every $\in > 0$, an $s \in S$ exists with $\|s - \inf(S)\| < \in$. (See 4 / Chapter 1 on the last homework set.) If $\inf(S) \in S$, we are done; otherwise the above condition means precisely that $\inf(S)$ is an accumulation point of S.
Now if S is closed, it must contain its accumulation points, and again we conclude that $\inf(S) \in S$. The proof is complete.

5 10 / Chapter 2

5.1 (a)

The statement is false.
int(cl(Q))=int(R)=R but int(Q)=∅

5.2 (b)

For cl(A)∩A=A, it is sufficient (why?) to show that A⊂cl(A). But cl(A) is defined as an intersection of sets, each of which contains A, and we're done. True.

5.3 (c)

The statement is false.
Take any open set A. cl(int(A)), as any closure, is a closed set, so cl(int(A))=A implies than any open set is closed, which is clearly wrong.

5.4 (d)

The statement is false.
bd(cl(Q))=bd(R)=∅ but bd(Q)=R.

5.5 (e)

A open $\Longrightarrow$A^C closed $\Longrightarrow$cl(A^C) =A^C $\Longrightarrow$bd(A)=cl(A)∩cl(A^C)=cl(A)∩A^C and bd(A)⊂A^C as desired. The statement is true.

6 12 / Chapter 2

6.1 (a)

int(any open set A)=A by definition: it is the union of all open subsets of A, namely the union of A (since it's open) and its proper open subsets, i.e. A itself.

Thus, since int(A) is open (union of open sets, by definition), int(B)=int(int(B)) for any set B.

6.2 (b)

int(A) is an open subset of A and int(B) is an open subset of B. So int(A)∪int(B) is an open subset of A∪B, so it is contained in int(A∪B), which is the union of all open subsets of A∪B.

6.3 (c)

int(A)∩int(B) is an open subset of A∩B, so by definition it is contained in int(A∩B).
int(A∩B) is an open subset of A∩B, so it is also an open subset of A, so it's contained in int(A). An analogous argument shows int(A∩B)⊂int(B).
So int(A)∩int(B)⊂int(A∩B) and int(A∩B)⊂int(A)∩int(B) and we're done.

7 28 / Chapter 2

7.1 (a)

Consider Z. It is infinite, but has no accumulation points, since any two distinct elements of Z are at least a distance 1 apart.

7.2 (b)

Take Q, Acc(Q)=R. Any interval of positive length will also work.

7.3 (c)

Consider $A = \{(\frac{1}{m}, \frac{1}{n}), m, n \in N\}$. $Acc(A) = \{(0,0)\} \cup \{(0, \frac{1}{m}), m \in N\} \cup \{(\frac{1}{m}, 0), m \in N\}$ is infinite and does not intersect A.

7.4 (d)

For $cl(A) = bd(A) = cl(A) - int(A)$, clearly any set with empty interior will do. (e),(f),(h) in Problem 2 / Chapter 2 serve as examples.