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Problem Set 12 Solutions
May 5, 2002

1 1

Say $f, g \in M^{\perp}$, $h \in M$ and $a \in R$. Then $< af + g, h >= a < f, h > + < g, h >= 0 \Rightarrow af + g \in M^{\perp}$ and thus $M^{\perp}$ is a space.

For completeness, say $(\forall n) f_n \in M^{\perp}$ and $f_n \to f$. Then:

$$< f, h >=< (f - f_n) + f_n, h >=< (f - f_n), h > + < f_n, h >=< (f - f_n), h >\leq \sqrt{\|f - f_n\| \|h\|} \to 0$$

so $< f, h >= 0$ and $f \in M^{\perp}$.

2 2

First, because the Legendre polynomials are obtained by applying the Gram-Schmidt algorithm to the set of functions $\{1, x, x^2, x^3, ...\}$, the set of all polynomials is clearly spanned by the Legendre polynomials. So it suffices to show that the polynomials are dense in $L^2([-1, 1]) = L^2$.

Now by Lemma 2 / p. 624 the set of periodic continuous functions $C_p([-1, 1]) = C$ is dense in L^2. So it suffices to show that the set of polynomials is dense in C (under the $\|\|_2$ norm).

Because the polynomials form an algebra, given any $f \in C$ and $\epsilon > 0$, by the Stone-Weierstrass there exists a polynomial p such that $\|f - p\| < \frac{\epsilon}{\sqrt{2}}$. But then:

$$(\|f - p\|_2)^2 = \int_{-1}^{1} |f(x) - p(x)|^2 dx \leq \int_{-1}^{1} \sup_{x \in [-1,1]} |f(x) - p(x)|^2 dx = \epsilon^2$$

so $\|f - p\|_2 \leq \epsilon$. Thus the set of polynomials is dense in C and the proof is complete.

3 3a

From Table 10.5-4, we know $g(x) = \frac{\sinh \pi}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{1-ik} e^{ikx}$ is the Fourier series for e^x.

Now e^x, periodically extended to all of R, has a discontinuity at $x = \pi$, with

$f(\pi^+) = e^\pi$ and $f(\pi^-) = e^{-\pi}$. Thus (see Table 10.3-1 in book),

$$\begin{aligned} g(\pi) &= \frac{e^\pi + e^{-\pi}}{2} = \cosh\pi = \frac{\sinh\pi}{\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{1-ik}e^{ik\pi} = \frac{\sinh\pi}{\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{1-ik}(\cos k\pi + i\sin k\pi) = \\ &= \frac{\sinh\pi}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{1-ik} = \frac{\sinh\pi}{\pi} + \frac{\sinh\pi}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{1-ik}+\frac{1}{1+ik}\right) = \\ &= \frac{\sinh\pi}{\pi}\left(1+2\sum_{k=1}^{\infty}\frac{1}{1+k^2}\right) \Rightarrow \pi\coth\pi - 1 = 2\sum_{k=1}^{\infty}\frac{1}{1+k^2} \end{aligned}$$

as desired.

4 4

In any metric space, the triangle inequality says $\|f_n - f\| \le |\|f_n\| - \|f\||$, and both are non-negative, so if $\|f_n - f\| \to 0$ (i.e. $f_n \to f$), then $|\|f_n\| - \|f\|| \to 0$, i.e. $\|f_n\| \to \|f\|$, as desired.

The converse is clearly not true: roughly speaking, the $f's$ can be of the same sizes but different. As an example in $L^2([-\pi,\pi])$, take any f that is not even, and define $f_n(x) = f(-x)$. If necessary, draw a picture to see why this serves as an example.

5 6

$<x,y>=\sum_{i=1}^{\infty} x_i y_i$. Then $<x,x>=\sum_{i=1}^{\infty} x_i^2 \ge 0$, because it is a sum of squares of real numbers, and $<x,x>=\sum_{i=1}^{\infty} x_i^2 = 0 \Leftrightarrow (\forall i) x_i = 0 \Leftrightarrow x = 0$. Further, $<x,y+w>=\sum_{i=1}^{\infty} x_i(y_i+w_i) = \sum_{i=1}^{\infty} x_i y_i + \sum_{i=1}^{\infty} x_i w_i =<x,y>+<x,w>$, $<\alpha x,y>=\sum_{i=1}^{\infty}\alpha x_i y_i = \alpha\sum_{i=1}^{\infty} x_i y_i = \alpha <x,y>$ and, last, $<x,y>=\sum_{i=1}^{\infty} x_i y_i = \sum_{i=1}^{\infty} y_i x_i =<y,x>$. Thus the given definition satisfies the requirements for an inner product.

For completeness, say $x_n = (x_{n1}, x_{n2}, ...)$ is Cauchy. Then $\{x_{nk}\}_{n=1}^{\infty}$ is also Cauchy, and all its elements are real, so $\lim_{n\to\infty} x_{nk} = x_k^*$ exists for all k. Then $x = (x_1^*, x_2^*, ...)$ satisfies the conditions for being $\lim_{n\to\infty} x_n$ and l_2 is complete, as claimed.

6 8

Set $f_n(x) = \{ \begin{array}{l} 0 \text{ for } x \in [-1,0] \\ \sqrt{4n^2x} \text{ for } x \in [0, \frac{1}{2n}] \\ \sqrt{4n-4n^2x} \text{ for } x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0 \text{ for } x \in [\frac{1}{n}, 1] \end{array}$. Then $f(x) = \lim_{n\to\infty} f_n(x) = 0$ for all $x \in [-1,1]$, but in mean $(\|f_n - f\|_2)^2 = (\|f_n\|_2)^2 = 1$ for all n, so f_n does not converge to the pointwise limit f in mean.

7 22

By Worked Example 10.2.,

$$< \ f, g >= \lim_{n\to\infty} < \sum_{k=0}^{n} < f, \varphi_k > \varphi_k, \sum_{j=0}^{n} < g, \varphi_j > \varphi_j >= \lim_{n\to\infty} \sum_{k,j=0}^{n} < f, \varphi_k > \overline{< g, \varphi_j >} < \varphi_k, \varphi_j >=$$

$$= \lim_{n\to\infty} \sum_{k,j=0}^{n} < f, \varphi_k >< \varphi_j, g > \delta_{jk} = \lim_{n\to\infty} \sum_{k=0}^{n} < f, \varphi_k >< \varphi_k, g >= \sum_{k=0}^{\infty} < f, \varphi_k >< \varphi_k, g >$$

8 30

$\widehat{\delta'}(k) = \frac{1}{2\pi}\int_R \delta'(x)e^{-ikx}dx = -\frac{1}{2\pi}\frac{d}{dx}(e^{-ikx})|_{x=0} = \frac{ik}{2\pi}$.

$\widehat{\delta}(k) = \frac{1}{2\pi}\int_R \delta(x)e^{-ikx}dx = \frac{1}{2\pi}e^{-ikx}|_{x=0} = \frac{1}{2\pi}$

so, as expected, $\widehat{\delta'}(k) = ik\widehat{\delta}(k)$.